UNIVERSITY of WASHINGTON

NSF

Acoustic scattering in unfriendly domains

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Householder Symposium XXII

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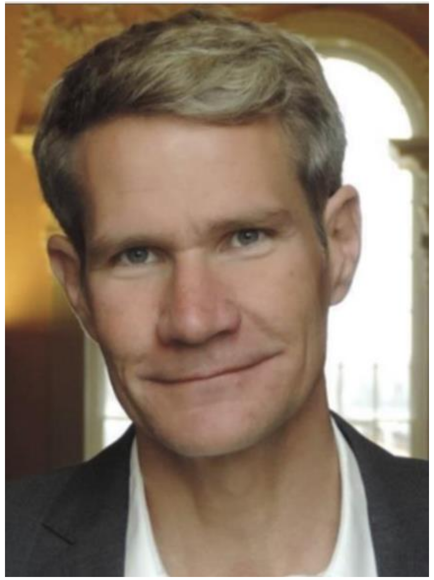
In collaboration with…

Wietse Vaes
Univ. of Washington

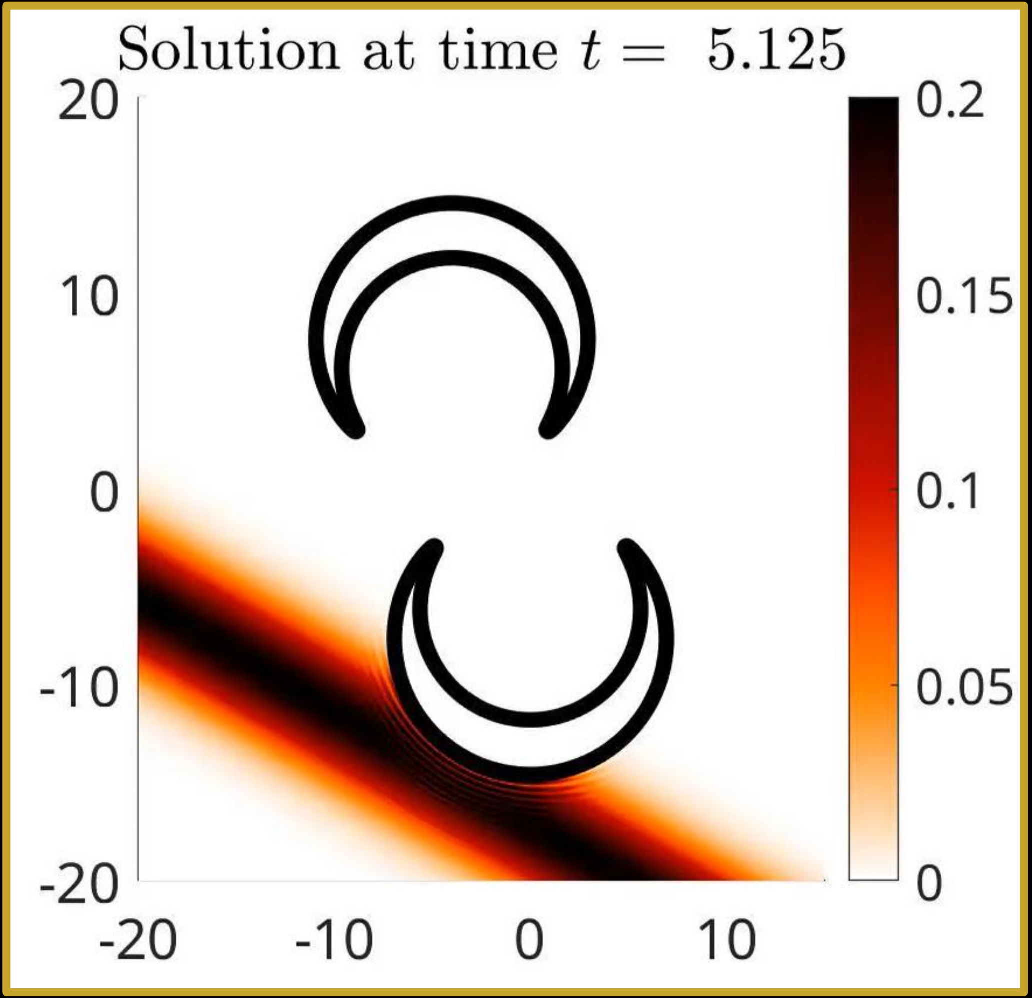


Per-Gunnar Martinsson
UT Austin

Abinand Gopal
UC Davis

Special thanks to Vladimir Rokhlin at Yale for visits and conversations.

Acoustic scattering



$u_{inc}(x,t) :=$ Incident field

$u(x,t) :=$ Scattered field

$u(x,t) + u_{inc}(x,t) :=$ total field

$$\frac{\partial^2 u}{\partial t^2}(x,t) - c^2 \Delta u(x,t) = 0, \quad x \in \Omega,$$
$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0, \quad x \in \Omega,$$
$$u(x,t) = -u_{inc}(x,t), \quad (x,t) \in \partial\Omega \times [0,T].$$

A numerical challenge…

Fast transform methods

- nonuniform FFTs
- bandlimited functions

Quadrature methods

- highly oscillatory integrals
- handling (near) singular integrands

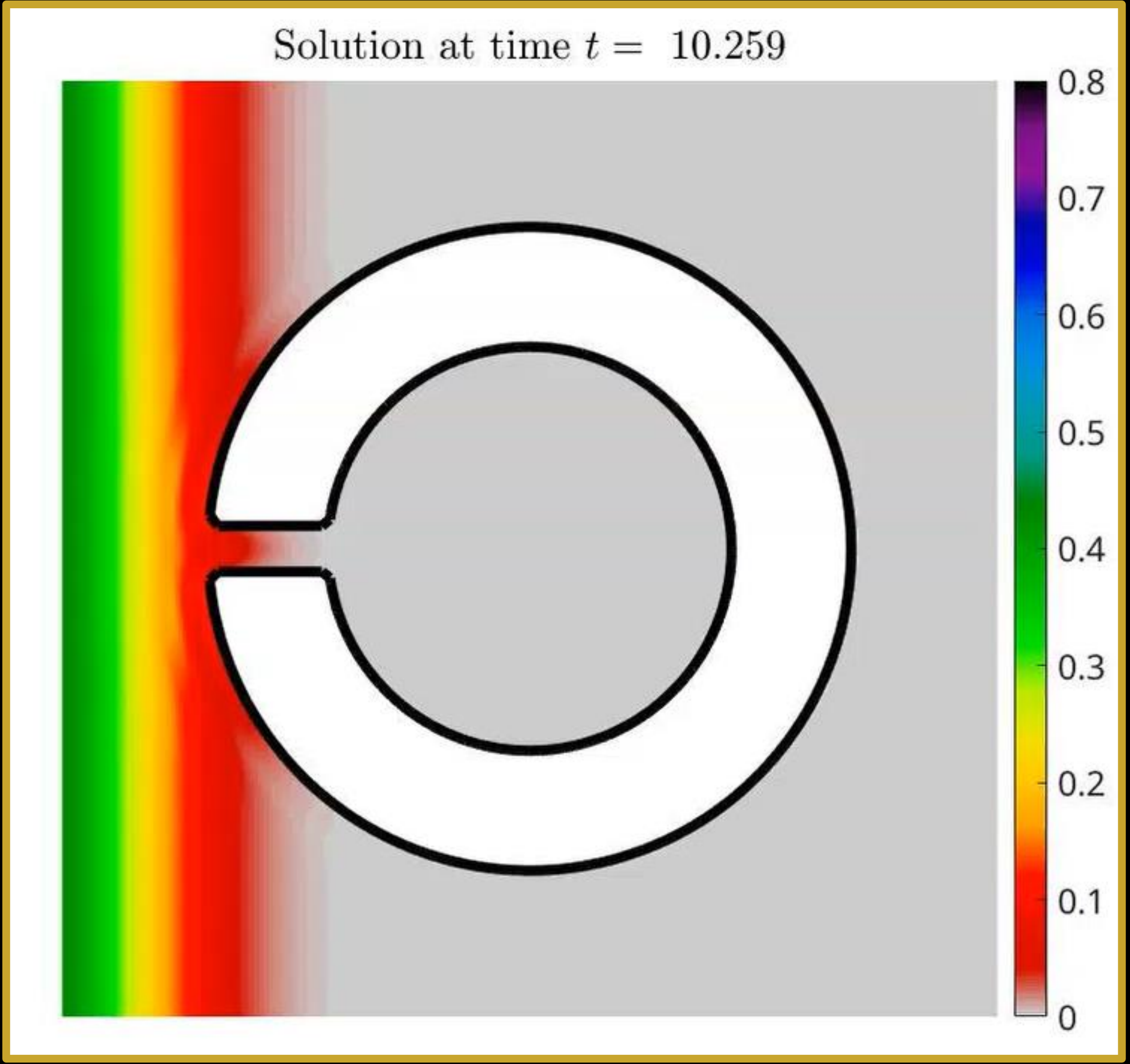
Approximation theory

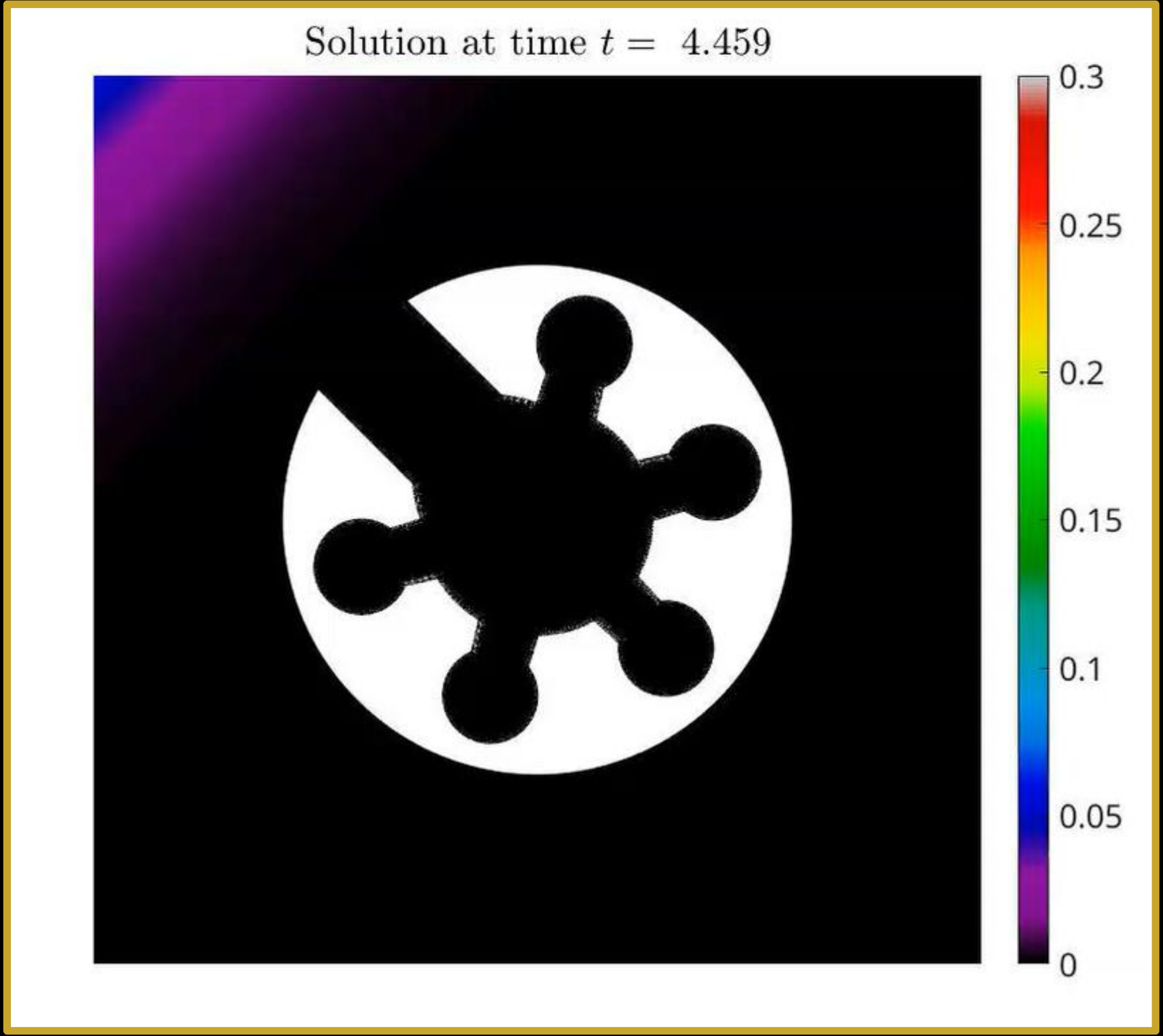
- multipole expansions
- functions near singularities

Boundary integral equations

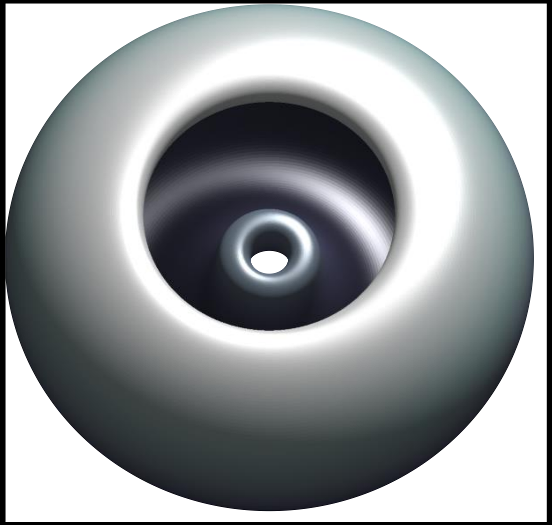
- contour integration methods
- fast direct solvers
- parameter-dependent PDEs

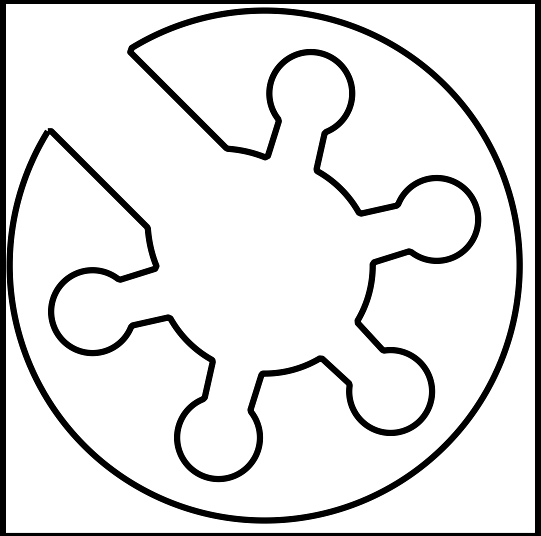
Unfriendly domain features: trapping regions and corners





Traditional direct-in-time methods





- Spatial discretization must impose artificial (absorbing) boundary conditions
- No "time-skipping"
- Small timesteps
- Accumulation of dispersive error over time

Idea! Express solution in integral form using Fourier transform...

Hybrid time-frequency methods

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{U}(x,\omega)e^{-i\omega t}\mathrm{d}\omega$$

$$u(x,t) \approx \frac{1}{2\pi}\int_{W_1}^{W_2} \hat{U}(x,\omega)e^{-i\omega t}\mathrm{d}\omega$$

$$u(x,t) \approx \sum_{j=1}^{m} \eta_j \hat{U}(x,\omega_j)e^{-i\omega_j t}$$

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HIGH-ORDER, DISPERSIONLESS "FAST-HYBRID" WAVE EQUATION SOLVER. PART I: O(1) SAMPLING COST VIA INCIDENT-FIELD WINDOWING AND RECENTERING*

THOMAS G. ANDERSON†, OSCAR P. BRUNO†, AND MARK LYON‡

Abstract. This paper proposes a frequency/time hybrid integral-equation method for the time-dependent wave equation in two- and three-dimensional spatial domains. Relying on Fourier transformation in time, the method utilizes a fixed (time-independent) number of frequency-domain integral-equation solutions to evaluate, with superalgebraically small errors, time-domain solutions

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SOLUTION OF ACOUSTIC SCATTERING PROBLEMS BY MEANS OF SECOND KIND INTEGRAL EQUATIONS

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Exxon Production Research Company, PO Box 2189, Houston, TX 77001, USA

Received 7 October 1982

In the present paper, the problem of scattering of acoustic waves from a fluid inclusion in two dimensions is solved numerically. The boundary conditions are those of continuous pressure and normal displacement. First, the problem in the

See also: rational approximation methods (Bruno and Santana, 2025, Bruno, Santana and Trefethen, 2024), convolutional quadrature (Lubich, 1994, Banjai, 2012, and many more), Mechocci, Misici, Recchioni, and Zirilli (2000), retarded layer potentials and time integration (Ha-Doung, 2003, Barnett, Greengard, and Hagstrom, 2020, many more)

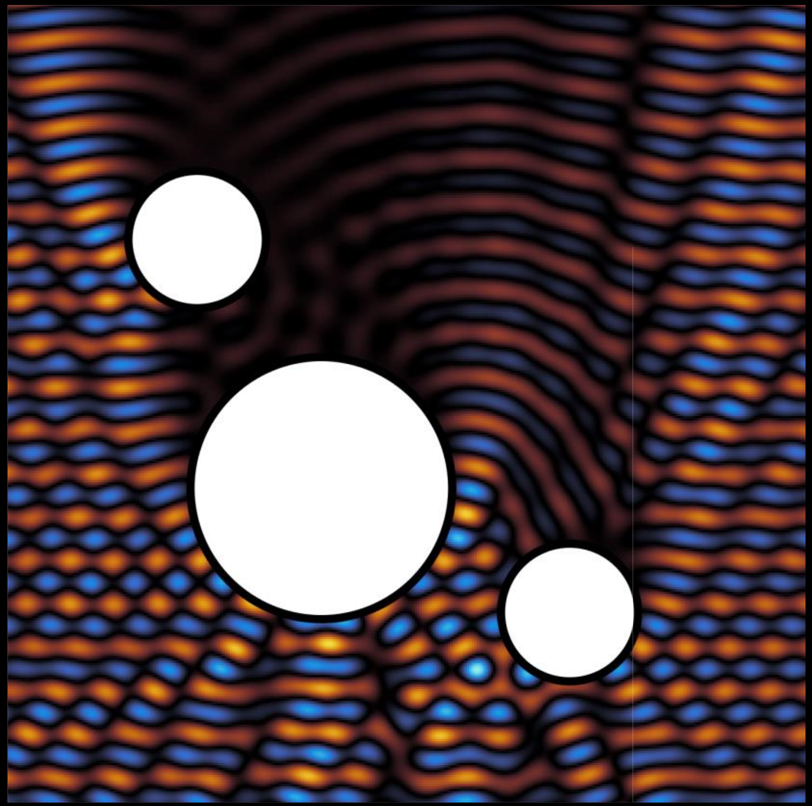
A boundary integral approach

$$\Delta \hat{U}(x,\omega) + \frac{\omega^2}{c^2}\hat{U}(x,\omega) = 0, \quad x \in \Omega,$$
$$\hat{U}(x,\omega) = -\hat{U}_{inc}(x,\omega), \quad x \in \partial\Omega,$$
$$+ \text{ Sommerfeld radiation condition}$$

$$\hat{U}(x,\omega_\ell) = \int_{\partial\Omega} \left(d_{\kappa_\ell}(x,y) + i\kappa_\ell s_{\kappa_\ell}(x,y)\right) \phi_\ell(y) \, \mathrm{ds}(y),$$

unknown

$$s_{\kappa_\ell}(x,y) = G_{\kappa_\ell}(x-y),$$
$$d_{\kappa_\ell}(x,y) = \vec{n}(x) \cdot \nabla_x G_{\kappa_\ell}(x-y),$$
$$G_{\kappa_\ell}(x) = \frac{i}{4} H_0^{(1)}(\kappa_\ell |x|).$$



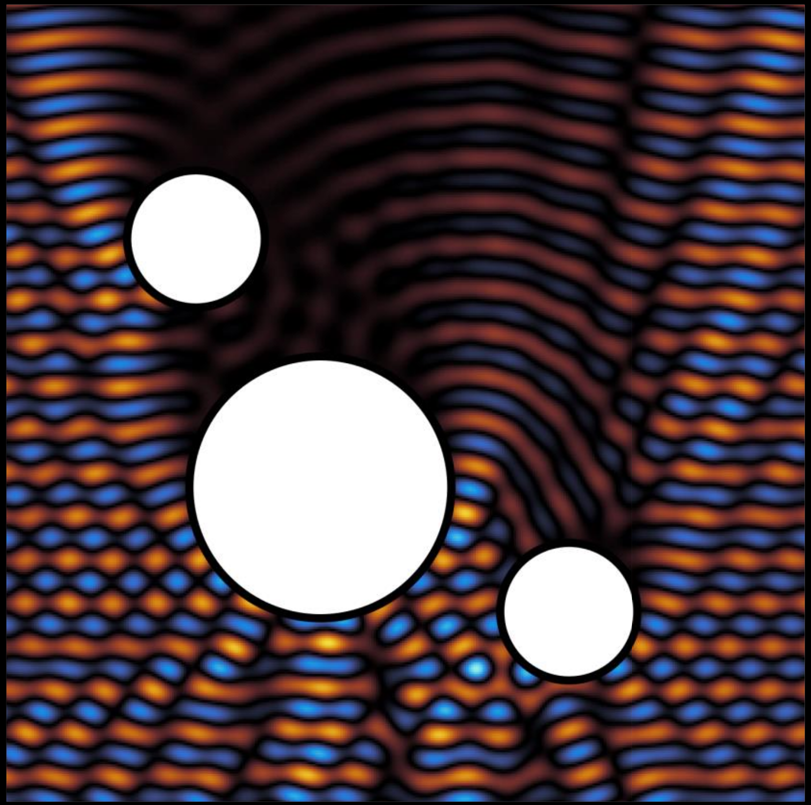
A boundary integral approach

$$\Delta \hat{U}(x,\omega) + \frac{\omega^2}{c^2}\hat{U}(x,\omega) = 0, \quad x \in \Omega,$$
$$\hat{U}(x,\omega) = -\hat{U}_{inc}(x,\omega), \quad x \in \partial\Omega,$$
$$+ \text{ Sommerfeld radiation condition}$$

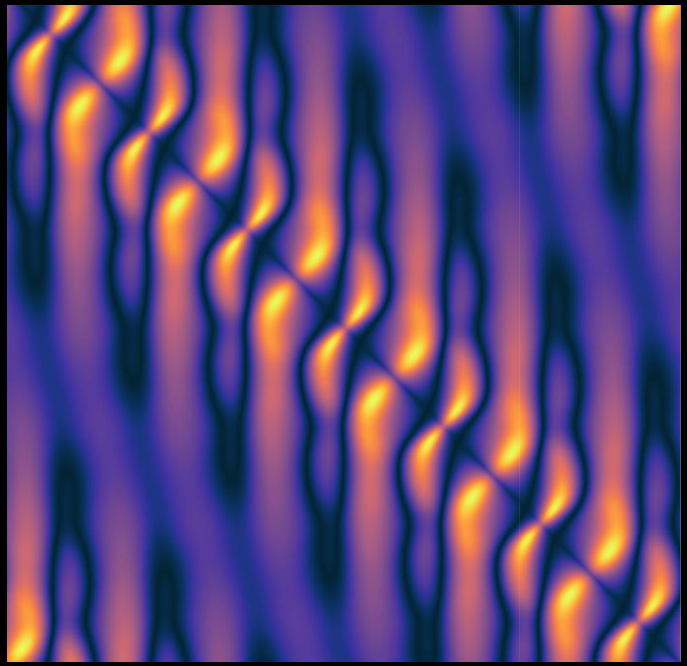
1. Solve for $\hat{U}(\cdot,\omega_\ell)$ for each ω_ℓ via boundary integral methods.

$$\hat{U}(x,\omega_\ell) = \int_{\partial\Omega} \left(d_{\kappa_\ell}(x,y) + i\kappa_\ell s_{\kappa_\ell}(x,y)\right)\phi_\ell(y)\mathrm{ds}(y),$$

2. Evaluate $\hat{U}(x_j,\omega_\ell)$ at relevant x_j.



A boundary integral approach



$$\Delta \hat{U}(x,\omega) + \frac{\omega^2}{c^2}\hat{U}(x,\omega) = 0, \quad x \in \Omega,$$

$$\hat{U}(x,\omega) = -\hat{U}_{inc}(x,\omega), \quad x \in \partial\Omega,$$

$$+ \text{ Sommerfeld radiation condition}$$

$$\frac{1}{2}\phi_\ell(x) + \int_{\partial\Omega} \left(d_{\kappa_\ell}(x,y) + i\kappa_\ell s_{\kappa_\ell}(x,y)\right)\phi_\ell(y)\mathrm{ds}(y) = -\hat{U}_{inc}(x,\omega_\ell).$$

$$\left(\frac{1}{2}I + A\right)\vec{\phi}_\ell = \vec{b}_\ell$$

- Fast direct solvers
- Broadband solvers?

Fast solvers: Rokhlin (1983), Martinsson and Rokhlin (2005), Gillman Young, and Martinsson (2012), many more…
Broadband solvers: Gopal and Martinsson (2024)

There's no free lunch...

Goal:

For each (x,t) pair in $\{x_1, \cdots x_M\} \times \{t_1, \cdots, t_N\}$, we must evaluate

$$u(x,t) \approx \sum_{j=1}^{m} \eta_j \hat{U}(x,\omega_j) e^{-i\omega_j t}$$

Naively, $\mathcal{O}(MNm + MmP)$ operations, $P =$ cost of evaluating $\hat{U}(x_i, \omega_j)$.

With NUFFT-III, $\mathcal{O}(MmP + M(m+N))$ operations.

solving Helmholtz at each ω_j + evaluating solution integral at all relevant x. **Yikes!**

There's no free lunch...

1. Make evaluating $\hat{U}(x,\omega)$ as efficient as possible.
   - Fast solvers (e.g., recursive skeletonization)
   - Fast matvecs (e.g., FMM)
   - Broadband Helmholtz solvers...

2. Limit the number of required evaluations of $\hat{U}(x,\omega)$.

A question of quadrature

$$u(x,t) \approx \frac{1}{2\pi}\int_{W_1}^{W_2} \hat{U}(x,\omega)e^{-i\omega t}\mathrm{d}\omega$$

$$u(x,t) \approx \sum_{j=1}^{m} \eta_j \hat{U}(x,\omega_j)e^{-i\omega_j t}$$

Number GL of quadrature points grows linearly with t!

Handling a highly oscillatory integral

$$u(x,t) \approx \frac{1}{2\pi}\int_{W_1}^{W_2} \hat{U}(x,\omega)e^{-i\omega t}\mathrm{d}\omega$$

1. Approximate $\hat{U}$ on $[W_1, W_2]$ with a trigonometric polynomial:

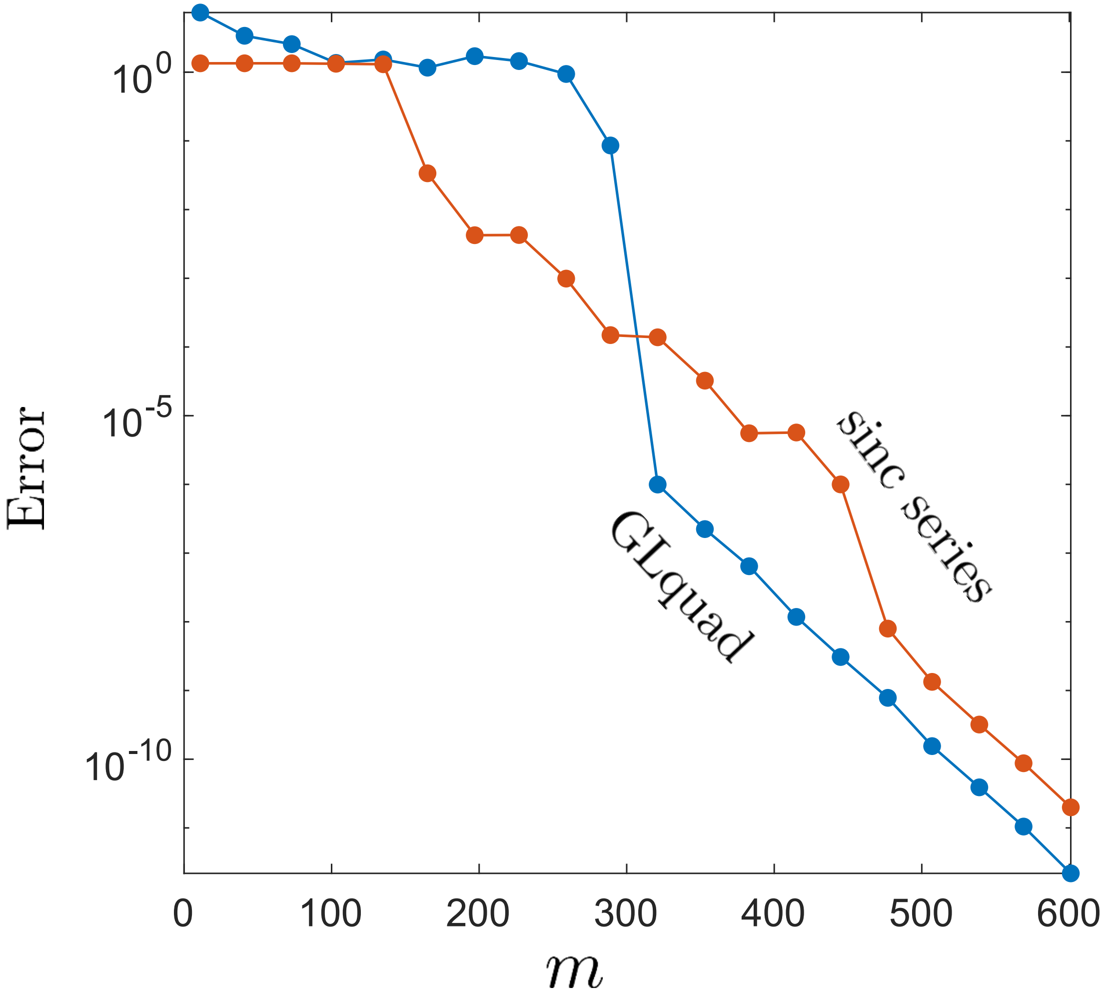
$$\hat{U}(x, Py + W_1) \approx \frac{1}{2m+1}\sum_{j=-m}^{m} c_j e^{2\pi i j y}, \quad \omega = Py + W_1, \quad P = W_2 - W_1.$$

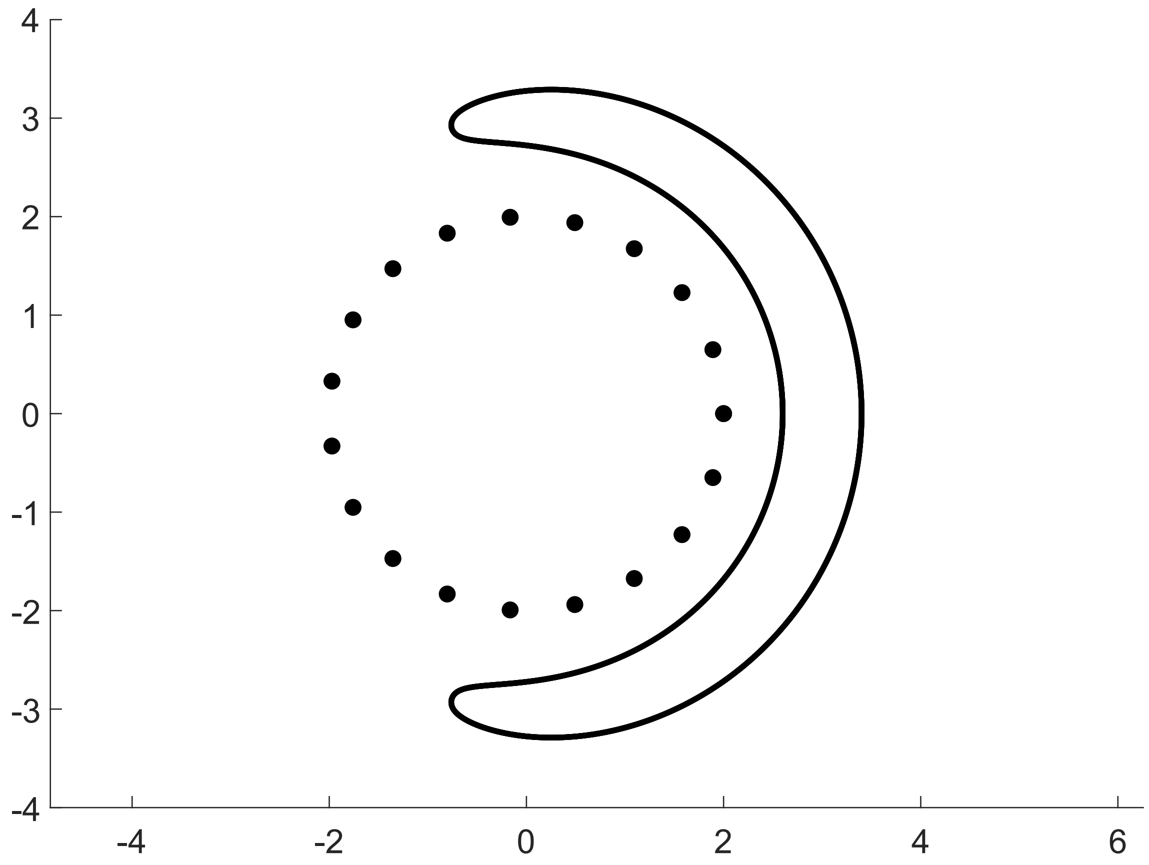
2. Substitute into integral and simplify:

$$u(x,t) \approx \frac{P}{2\pi(2m+1)} e^{-it(P/2+W_1)} \sum_{j=-m}^{m} (-1)^j c_j(x)\mathrm{sinc}\left(\frac{Pt}{2\pi} - j\right).$$

- Equivalent to Shannon Sampling Theorem for bandlimited functions + truncation (See Kirchies and Potts (2024))
- Equivalent to fast "scaled convolution" method (Anderson, Bruno, Lyon (2020))

The trouble with trapping...





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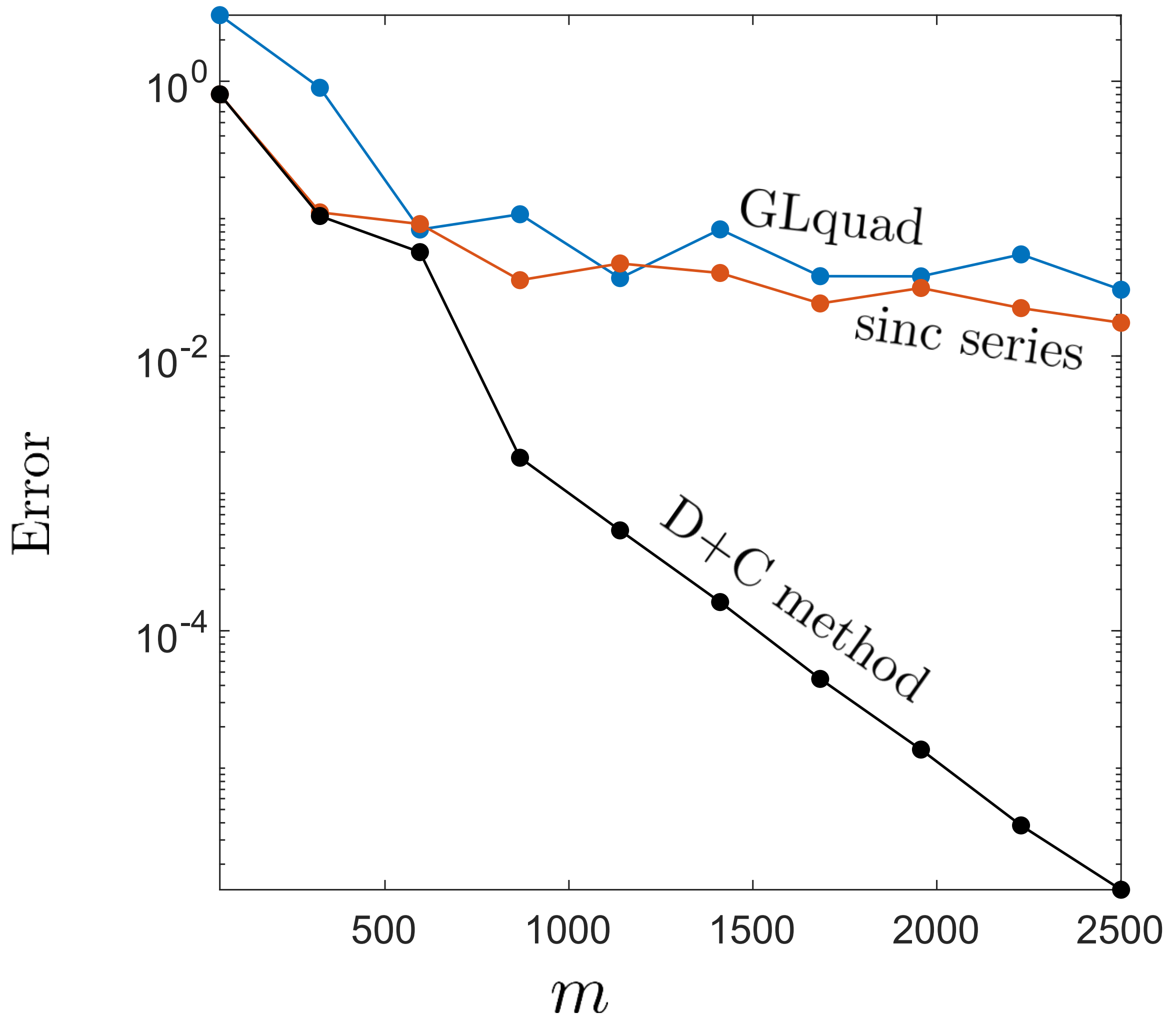
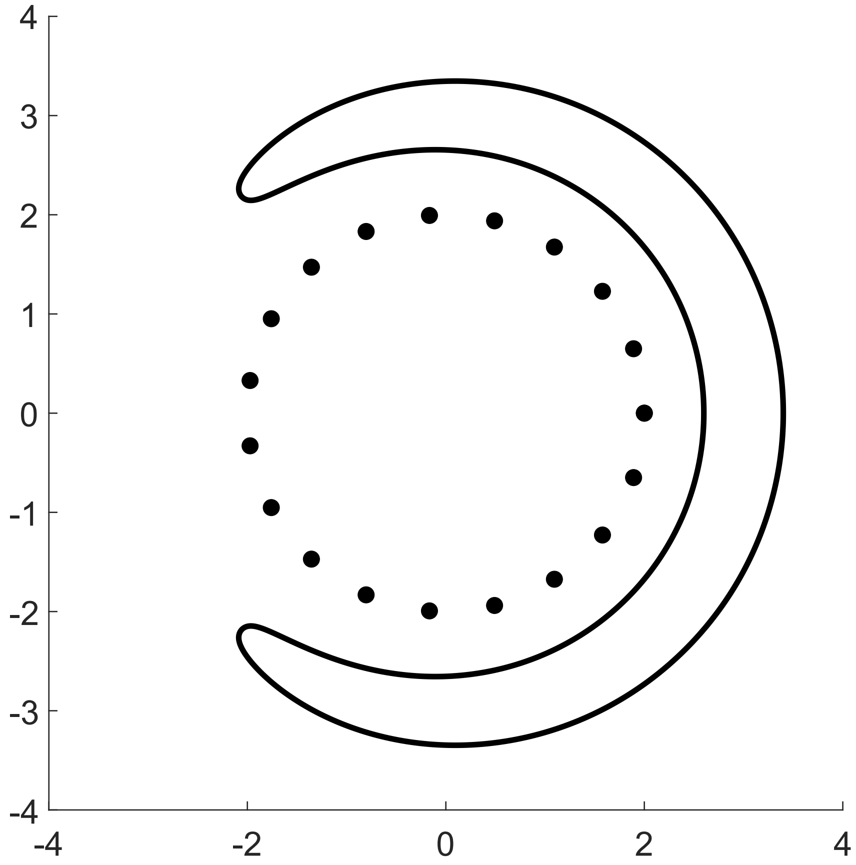
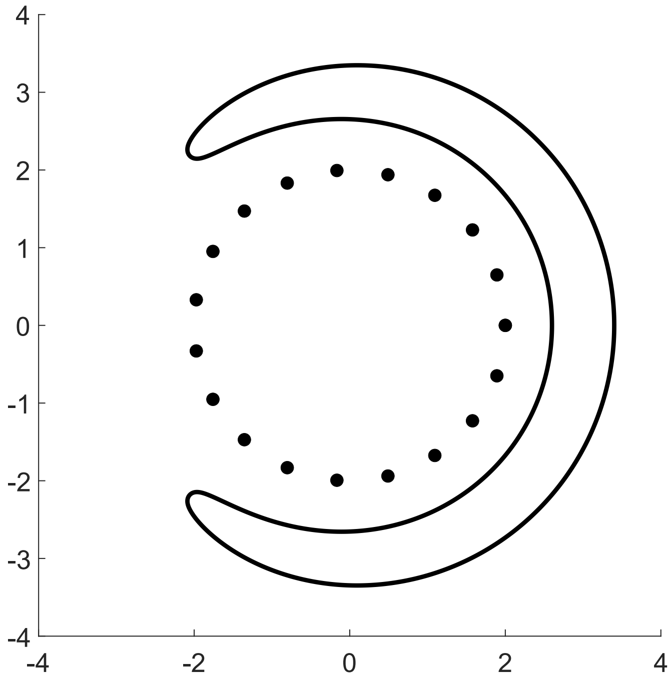
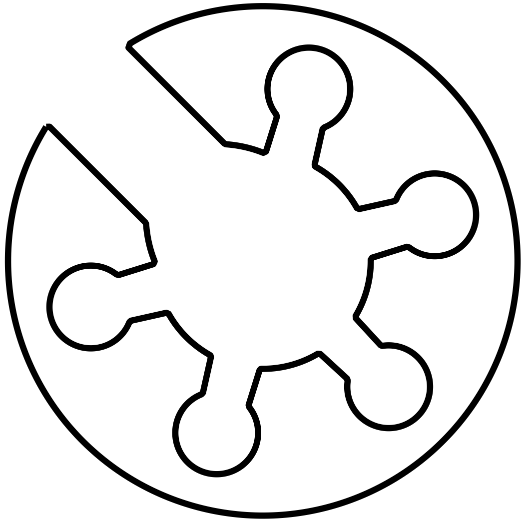




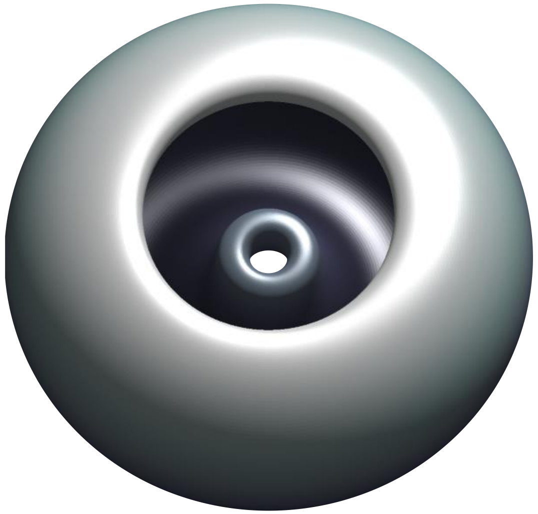
The trouble with trapping…



m > 5000



Not feasible



LOL!

T-dependency is inescapable

$$u_m(x,t) = \frac{P}{2\pi(2m+1)} e^{-it(P/2+W_1)} \sum_{j=-m}^{m} (-1)^j c_j(x) \mathrm{sinc}\left(\tfrac{Pt}{2\pi} - j\right).$$

Fact 1: If $\tilde{x}$ is in a friendly part of the domain, then

$$\|u(\tilde{x},\cdot) - u_m(\tilde{x},\cdot)\|_\infty \leq C\xi^{-m}$$

Fact 2: $c_j(\tilde{x}) \approx u\left(\tilde{x}, 2\pi j/(W_2 - W_1)\right)$

T-dependency is inescapable

$$u_m(x,t) = \frac{P}{2\pi(2m+1)} e^{-it(P/2+W_1)} \sum_{j=-m}^{m} (-1)^j c_j(x) \mathrm{sinc}\left(\tfrac{Pt}{2\pi} - j\right).$$

Unfortunate corollary:

$$\|u(\tilde{x},\cdot) - u_m(\tilde{x},\cdot)\|_\infty = \Theta(u(\tilde{x}, 2\pi m/P))$$

Approximation theorist

We need to improve analyticity properties!

Physicist

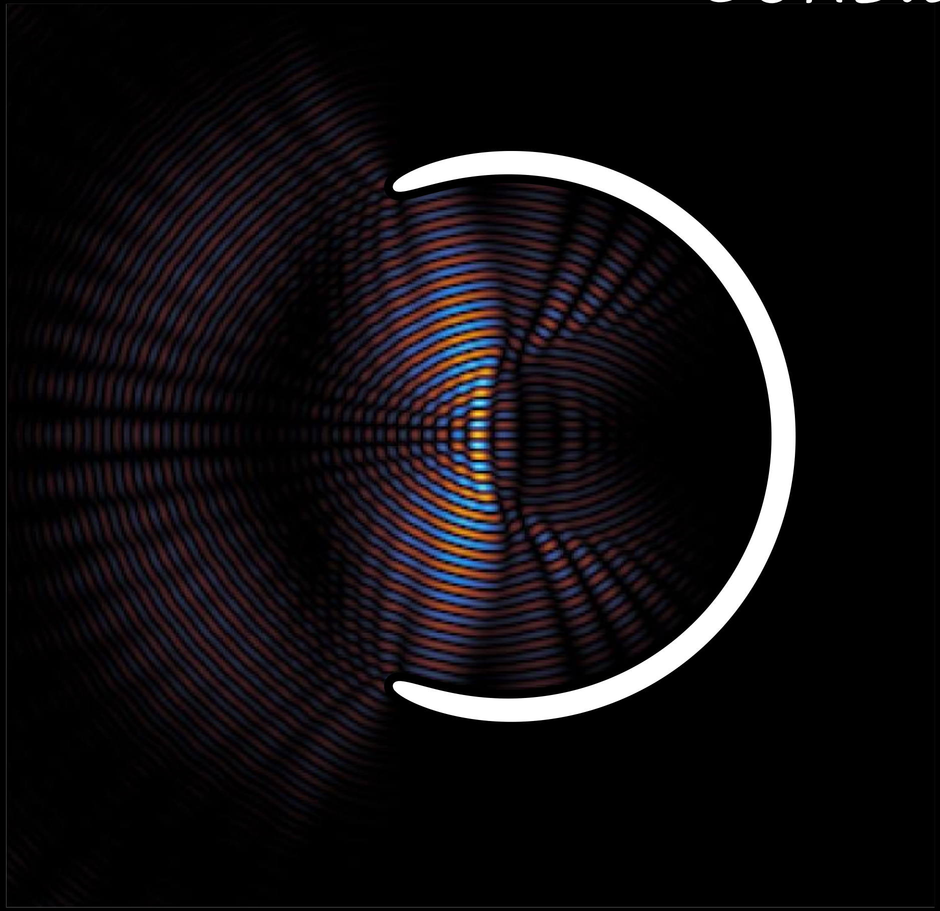
We need to identify and isolate/avoid slow-decaying eigenmodes!

Engineer

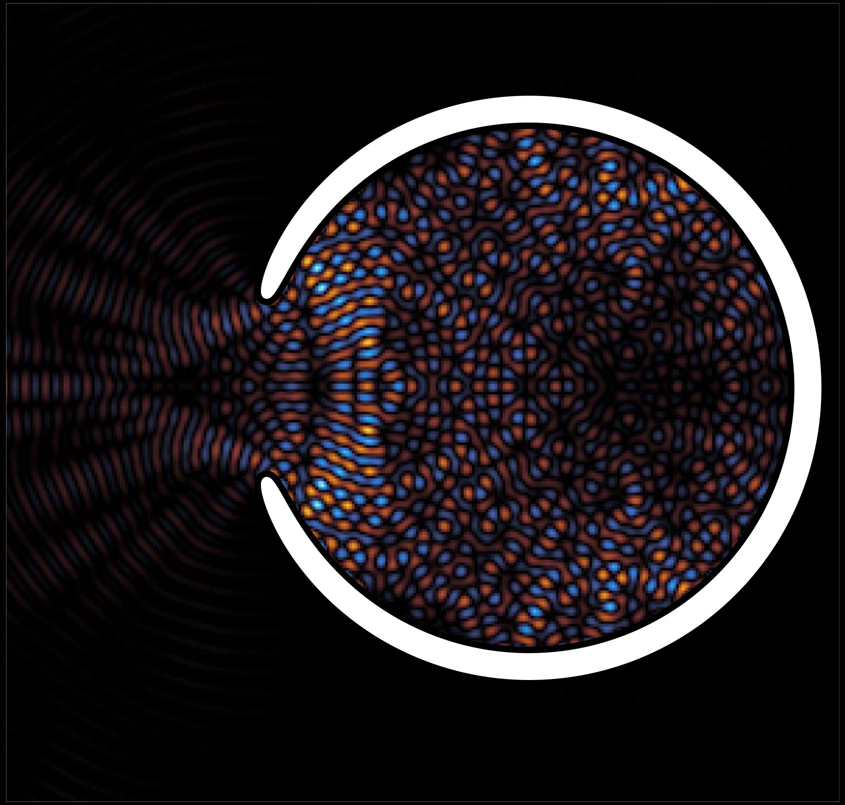
Damp!

The trouble with trapping

Consider two domains...



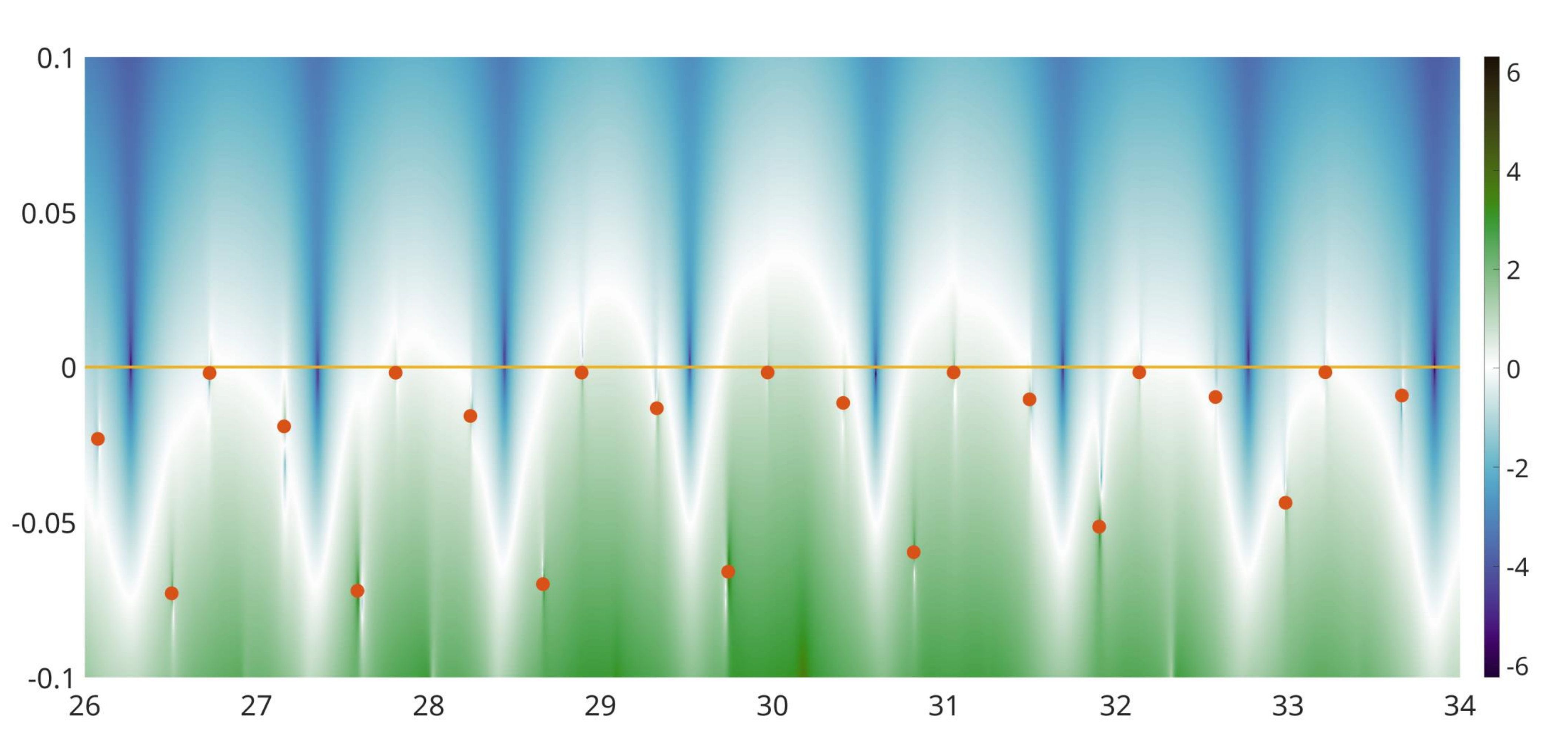
Real part of solution at $t = 46$.



Real part of solution at $t = 150$.

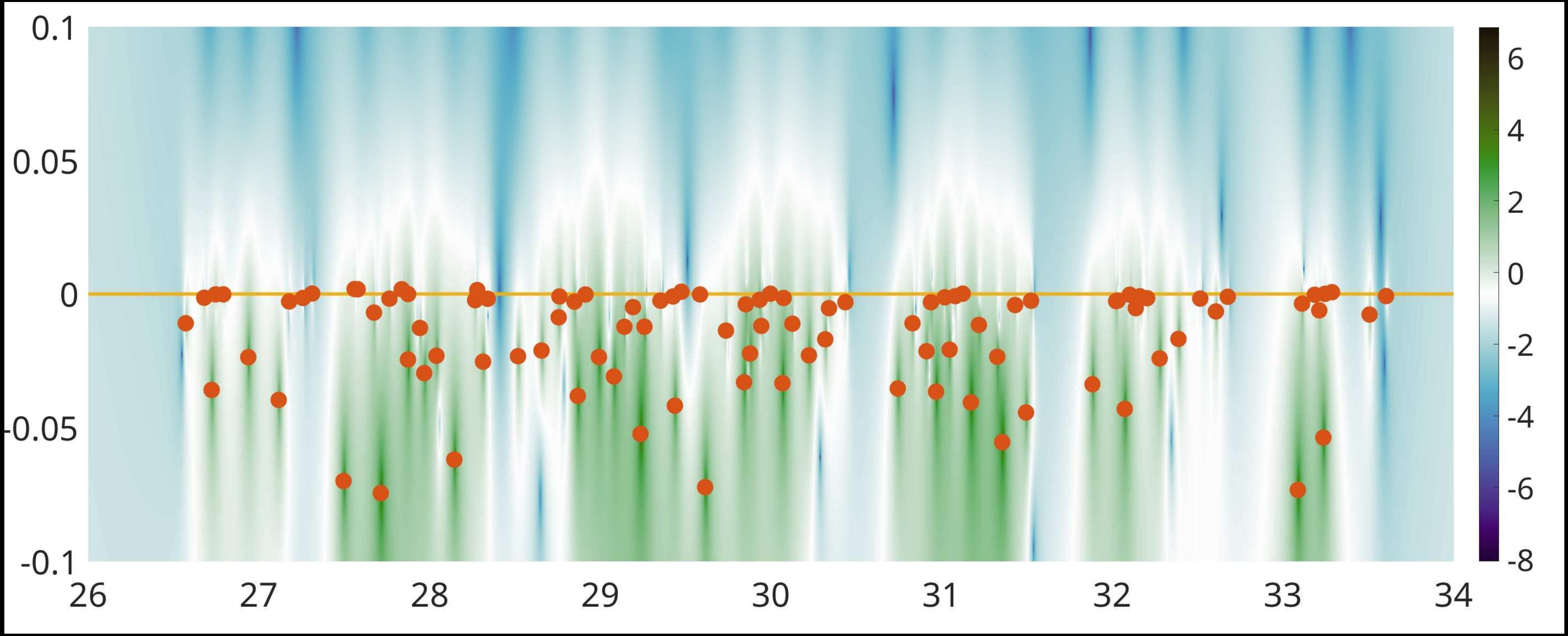
The trouble with trapping

Poles in the half arc case



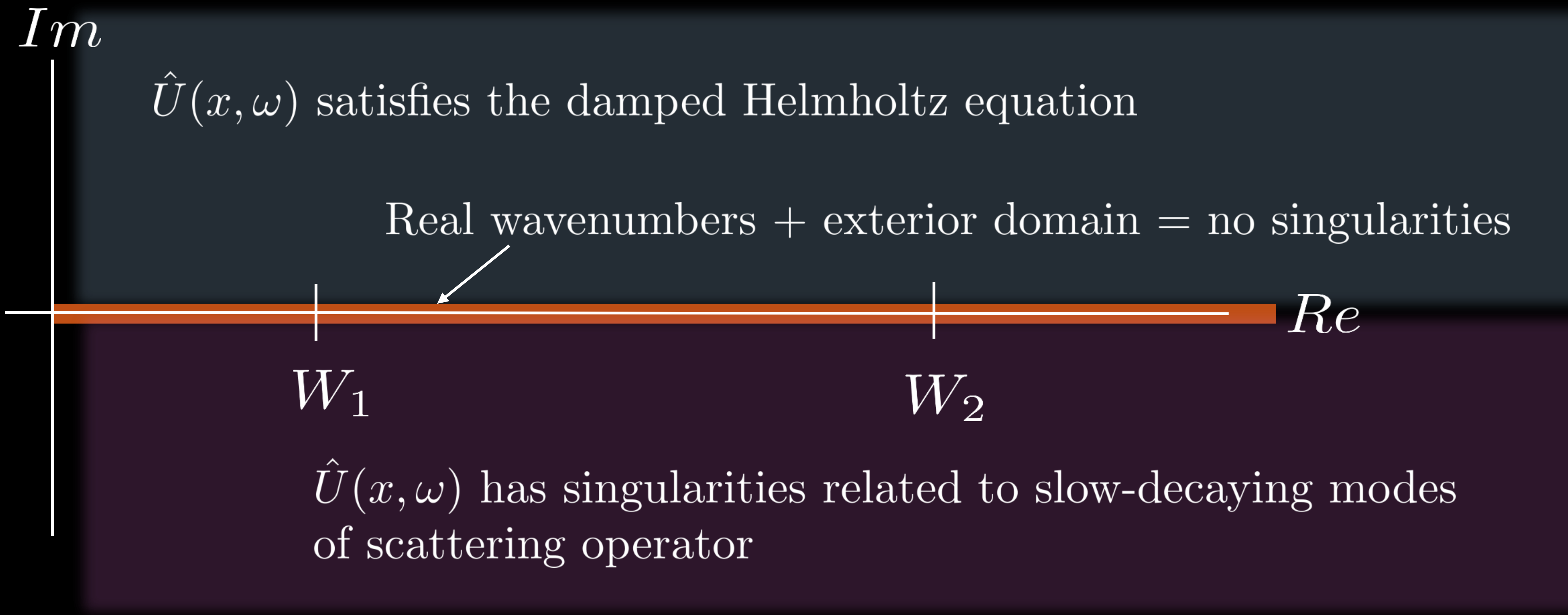
The trouble with trapping

Poles in the pinched arc case



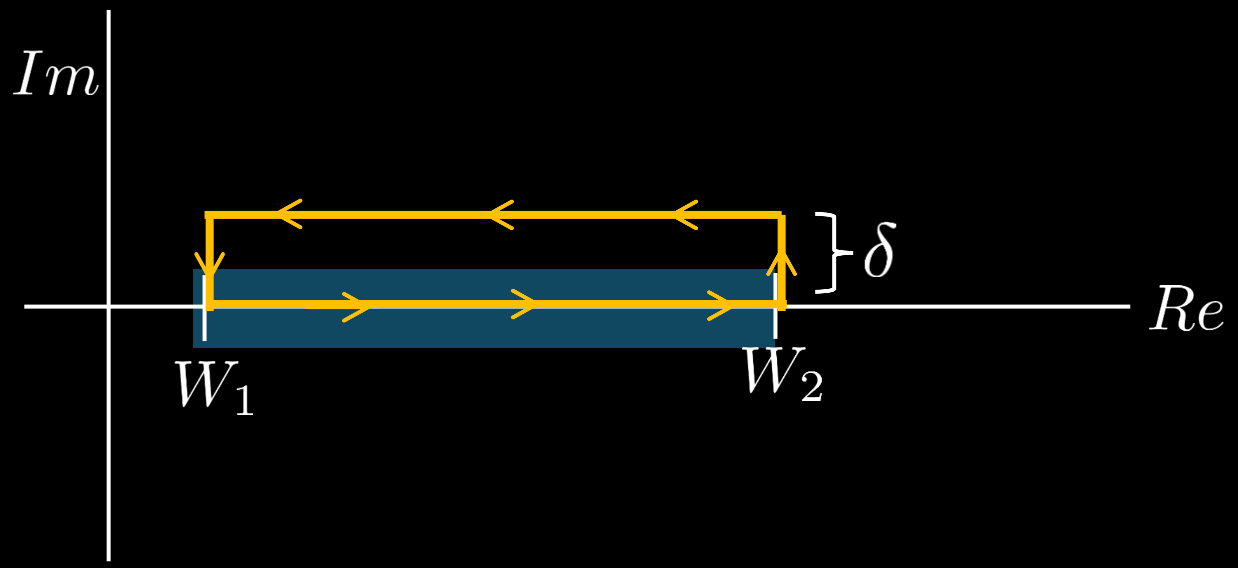
Scattering theory perspective

The complex ω-plane



Related: Bruno, Santana, and Trefethen (2024), Bruno and Santana (2025)

The complexification trick

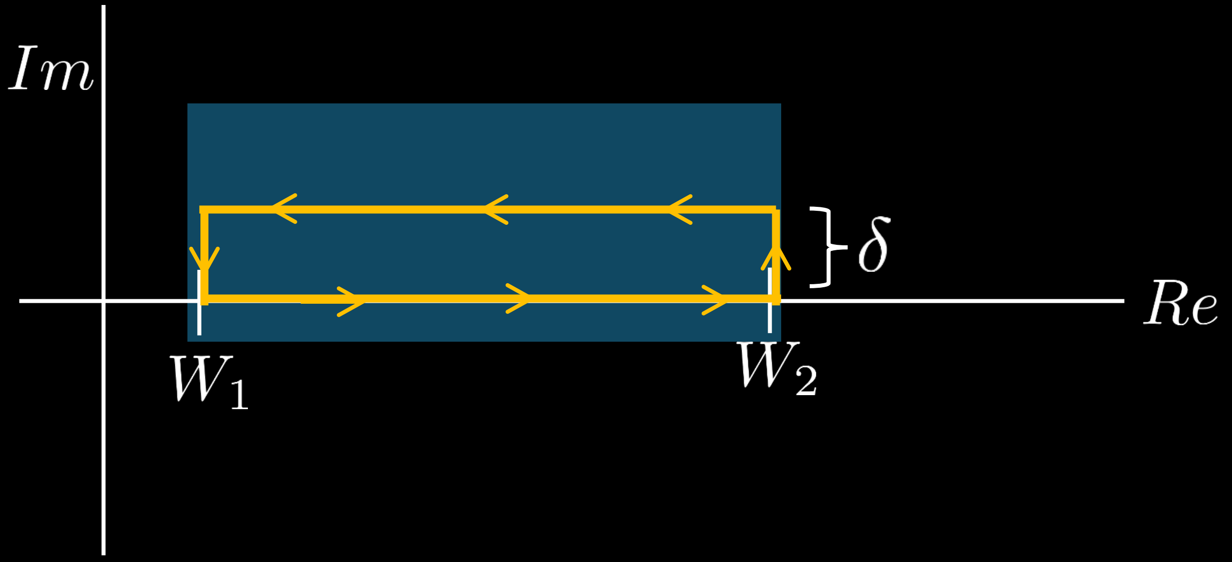


Undamped = Damped + corrections

$$\int_{W_1}^{W_2} \hat{U}(x,\omega)e^{-i\omega t}\mathrm{d}\omega = \underbrace{\int_{W_1}^{W_2} \hat{U}(x,\omega+\delta i)e^{-i(\omega+\delta i)t}\mathrm{d}\omega}_{I_\delta} - I_{cL} - I_{cR}$$

Rokhlin (1982)

The complexification trick



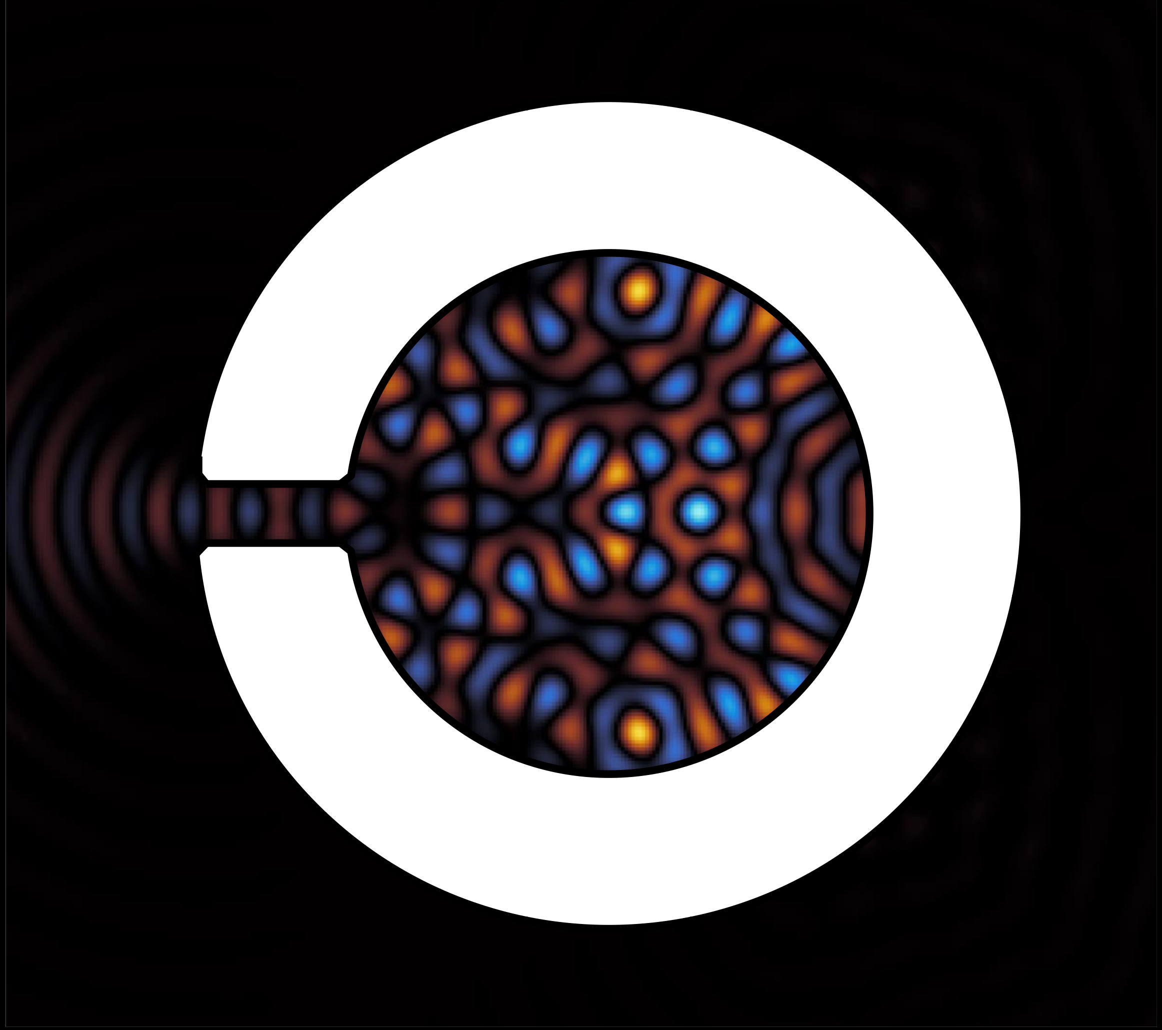
Undamped = Damped + corrections

$$\int_{W_1}^{W_2} \hat{U}(x,\omega)e^{-i\omega t}\mathrm{d}\omega = \underbrace{\int_{W_1}^{W_2} \hat{U}(x,\omega+\delta i)e^{-i(\omega+\delta i)t}\mathrm{d}\omega}_{I_\delta} - I_{cL} - I_{cR}$$

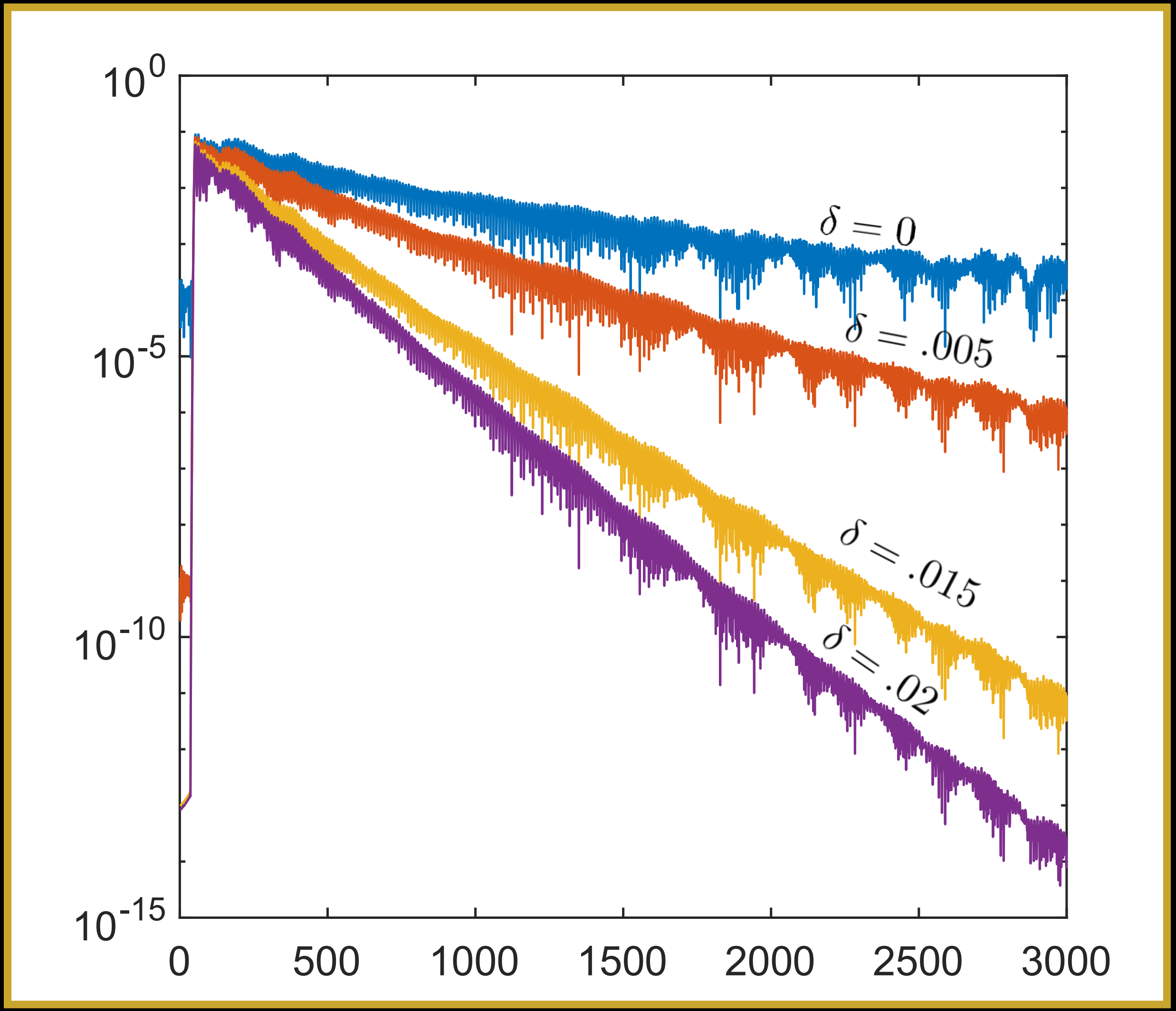
Rokhlin (1982)

The complexification trick

Real part of solution at $t = 130$



Magnitude of Fourier coefficients



Damped+corrected fast sinc quadrature scheme

1. Approximate $\hat{U}(x, \omega + \delta i)$ with a trigonometric polynomial $p_m(x)$:
   - $2m+1$ equispaced samples over $[W_1, W_2]$, FFT to get coefficients of $p_m(x)$.
     - Solve for densities at each ω_j.
     - Use integral form to evaluate each $\hat{U}(x, \omega_j)$ at all relevant $x \in \Omega$.

2. Integral I_δ simplifies so that

$$u(x,t) \approx \frac{P}{2\pi(2m+1)} e^{-it(P/2+W_1+i\delta)} \sum_{j=-m}^{m} (-1)^j c_j(x) \operatorname{sinc}\left(\tfrac{Pt}{2\pi} - j\right) - I_{cL} - I_{cR}.$$

3. Evaluate at all relevant (x, t) via fast sinc transform!

Fast sinc transform via FINUFFT library (H. Lawrence, Anderson, Barnett). See also Greengard, Lee, Inati (2006)

Thank you!

Summary: When you encounter problems…

Take refuge *in the complex plane, and* ***take advantage*** *of data sparsity!*

CORNELL

HOUSEHOLDER XXII