UNIVERSITY of WASHINGTON

Zolotarev numbers and the nonuniform discrete Fourier transform

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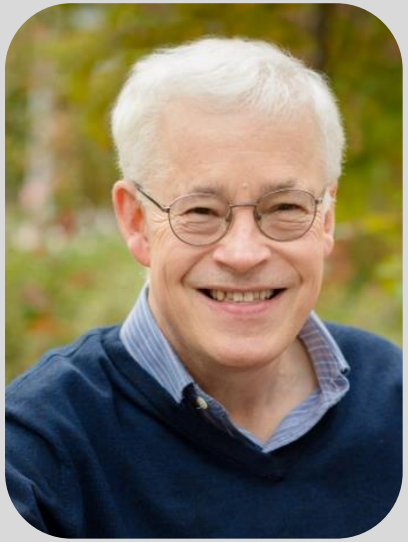
PACM Seminar, Princeton University

In collaboration with…

(E. Epperly, Caltech)

(A. Barnett, Flatiron Institute)



(L.N. Trefethen, Harvard)

(B. Beckermann, Univ. de Lille)

(D. Kressner, EPFL)

(A. Townsend, Cornell Univ.)

Today's focus: the nonuniform discrete Fourier transform

Solve $Vx = b$, where V is a Vandermonde matrix, with $\{\gamma_1, \ldots, \gamma_m\}$ on the unit circle.

$$V = \begin{pmatrix} (\gamma_1)^0 & (\gamma_1)^1 & \cdots & (\gamma_1)^{n-1} \\ (\gamma_2)^0 & (\gamma_2)^1 & \cdots & (\gamma_2)^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ (\gamma_m)^0 & (\gamma_m)^1 & \cdots & (\gamma_m)^{n-1} \end{pmatrix}$$

Related highly structured matrix families

Toeplitz, Hankel, Toeplitz+Hankel

Numerical PDEs

Covariance matrices in signal processing

Time series and dynamical systems

Function approximation

$$T = \begin{pmatrix} t_0 & t_{-1} & \cdots & \cdots & t_{-(n-1)} \\ t_1 & t_0 & \ddots & \ddots & t_{-(n-2)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & t_{-1} \\ t_{n-1} & \cdots & \cdots & t_1 & t_0 \end{pmatrix}$$

Cauchy-like, Loewner, Pick

Semi-separable function representation

Matrix equations

Dynamical systems

Rational approximation methods

$$C = \begin{pmatrix} \frac{1}{x_1 - y_1} & \frac{1}{x_1 - y_2} & \cdots & \frac{1}{x_1 - y_n} \\ \frac{1}{x_2 - y_1} & \frac{1}{x_2 - y_2} & \cdots & \frac{1}{x_2 - y_n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{1}{x_n - y_1} & \frac{1}{x_n - y_2} & \cdots & \frac{1}{x_n - y_n} \end{pmatrix}$$

What do all these matrices have in common?

1. They share a very special kind of *displacement structure,* enabling a *sparse representation.*

2. They are all just a *fast transform* away from *hierarchical low rank structure.*

Fast and superfast solvers

Example: Solve $n \times n$ linear system in as few operations as possible.

Fast

$\mathcal{O}(n^2)$

Levinson Algorithm [Levinson (1947) Durbin (1962)]

Schur Algorithm [Lev-Ari (1983), Kailath and Sayed (1991)]

Displacement-based GE [Heinig (1995), Gohberg, Kailath and Olshevsky(1995)]

Fast and superfast solvers

Example: Solve $n \times n$ linear system in as few operations as possible.

Superfast

$$\mathcal{O}(n \log^p(n) \log^q(1/\epsilon))$$

The square Toeplitz case

Heinig (1998)
Chandrasekaran, Gu, Xia, Zhu (2007)
Martinsson, Rokhlin, Tygert (2005)
Xia, Xi, Gu (2007)

Overdetermined Toeplitz system

Xi, Xia, Cauley, Balakrishnan (2014)

Can we adapt these ideas to Vandermonde systems, and beyond?

Templates for a superfast solvers

Example: Solve $n \times n$ linear system in as few operations as possible.

Template for superfast solver for $Ax = b$:

1) Apply fast transform: $A \rightarrow \hat{A}$, $b \rightarrow \hat{b}$
2) Compress rank-structured matrix: $\hat{A} \approx H$
3) Solve: $H\hat{x} = \hat{b}$
4) Transform solution: $\hat{x} \rightarrow x$

Displacement structure

+

Zolotarev rationals for low rank approximation

Zolotarev rationals and low rank approximation

The inverse nonuniform discrete Fourier transform

Zolotarev's rational approximation problems

Y. Zolotarev
(1847-1878)

Zolotarev's third problem:

Rational analogue to Chebyshev polynomial approximation problem

Zolotarev's fourth problem:

Rational analogue to polynomial filtering

Rational approximation to sign function on disjoint intervals, square root function on [β, 1].

[Acheiser (1901), Todd (1984), Gončar (1969), Istace &Thiran (1995)]

Zolotarev's rational approximation problems

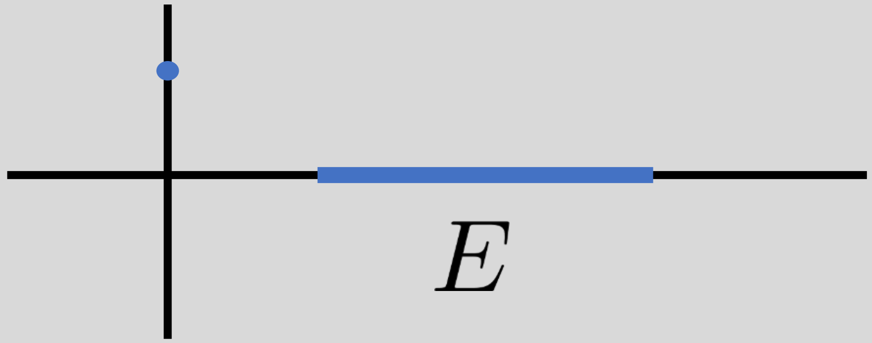
Y. Zolotarev
(1847-1878)

- Iterative solvers for matrix equations
- Low rank approximation and bounds on the decay of singular values of certain matrices
- Eigensolvers, polar decompositions, the SVD
- Tensor compression
- Nonlinear approximation schemes (e.g., via rationals or exponential sums)
- Spectral methods and PDE solvers
- Digital and analog filter design

[See Simoncini (2016) , Beckermann & Townsend (2019), Nakatsukasa & Freund (2016) for overviews]

Zolotarev's third problem

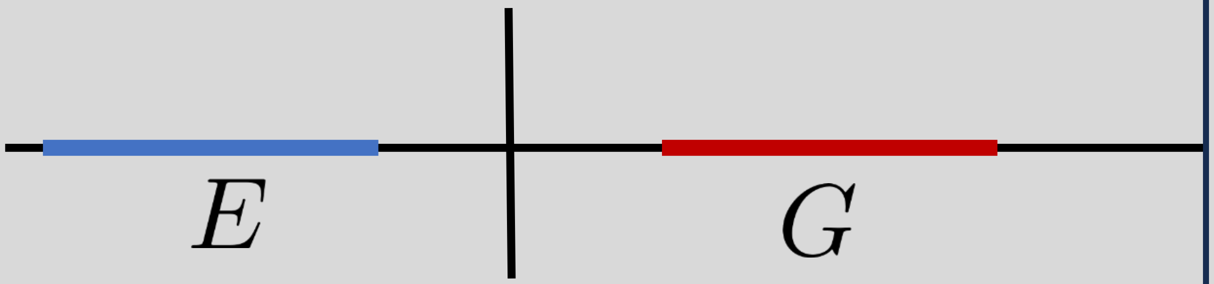
Chebyshev's problem



Find the polynomial $p(x)$ of degree $\leq k$ such that:

- $p(0) = 1$,
- $\max_{x \in G} |p(x)|$ is minimized.

Zolotarev's problem

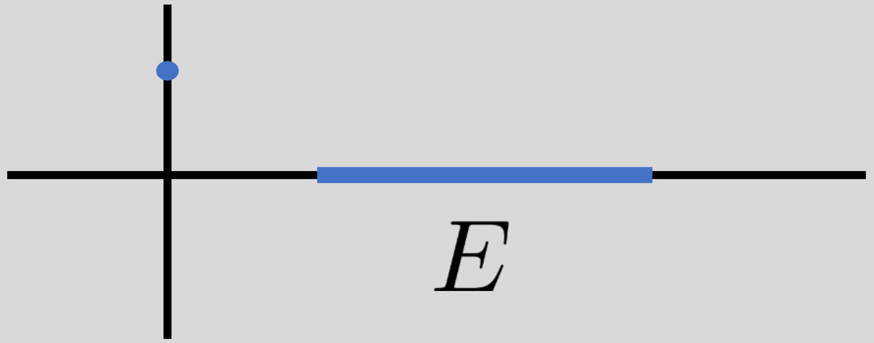


Find a rational $r(x)$ of type (k, k) such that:

- $\min_{x \in G} |r(x)| \geq 1$,
- $\max_{x \in E} |r(x)|$ is minimized.

Zolotarev's third problem

Chebyshev's problem



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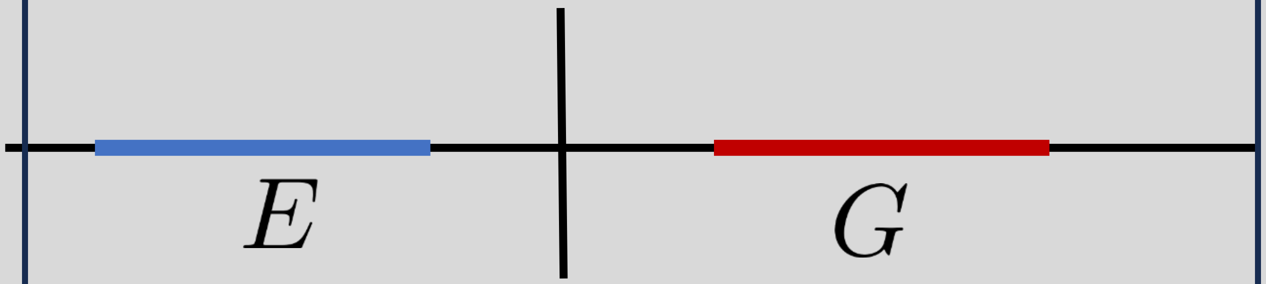
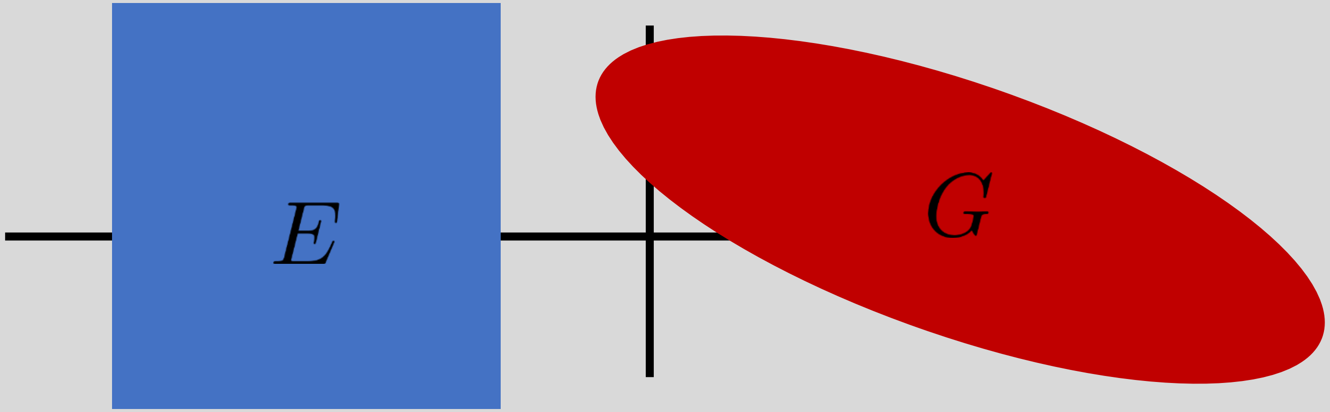


Find a rational $r(x)$ of type (k, k) such that:

$$\frac{\max_{x \in E} |r(x)|}{\min_{x \in G} |r(x)|}$$

is minimized.

Zolotarev's third problem



Find a rational $r(x)$ of type (k, k) such that the following is attained:

$$Z_k(E, G) := \min_{r \in \mathcal{R}^k} \frac{\max_{z \in E} |r(z)|}{\min_{z \in G} |r(z)|}.$$

$Z_k(E, G)$ is the kth Zolotarev number associated with $E \cup G$.

Known results: Disks and Intervals

- When E, G are intervals on $\mathbb{R}$, $r(z)$ is known exactly and can be expressed in terms of its poles and zeros, which are computed via elliptic integrals (Zolotarev, 1877).
- When E, G are disks in $\mathbb{C}$, $r(z)$ is known exactly and can be expressed in terms of a repeated pole and zero (Starke, 1992).
- $Z_k(E, G)$ is invariant under Möbius transformations.

$$h = \exp\left(\frac{1}{\mathrm{cap}(E,G)}\right)$$

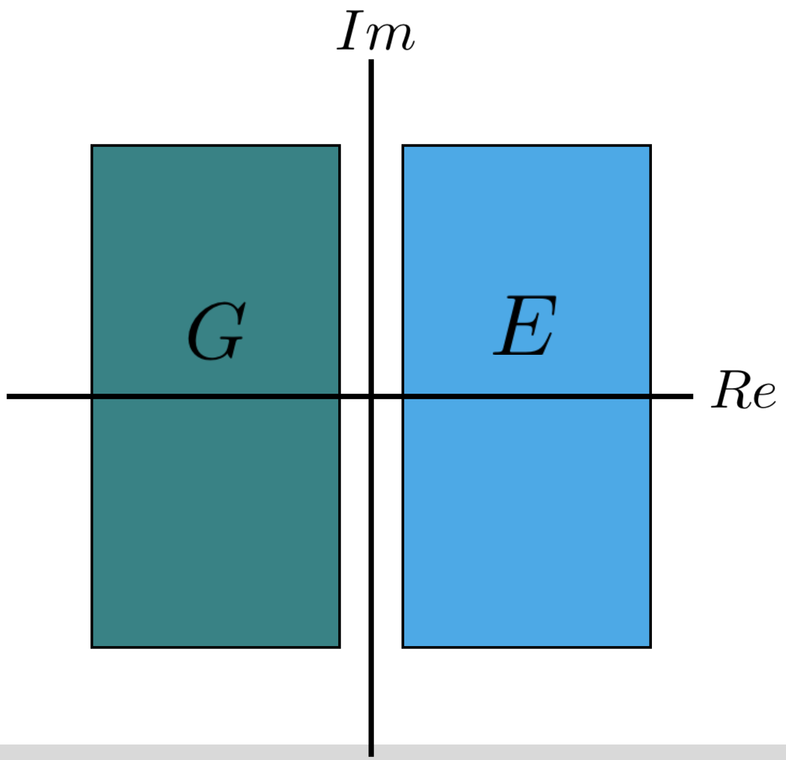
Disjoint sets E and G	Bound	Reference
finite intervals of $\mathbb{R}$	$Z_k(E,G) \leq 4h^{-k}$	Townsend & Beckermann (2016)
disks in $\mathbb{C}$	$Z_k(E,G) \leq h^{-k}$	Starke (1992)
arcs on a circle $\mathbb{C}$	$Z_k(E,G) \leq 4h^{-k}$	Useful for NUDFT (W., Epperly, Barnett, 2024)

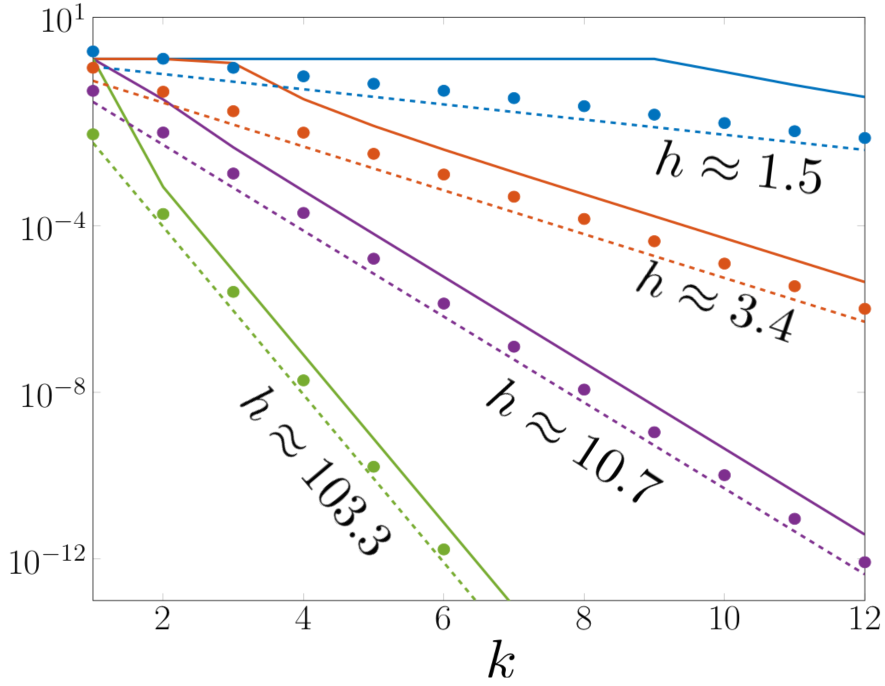
Known results: More general sets

Theorem (Rubin, Townsend, W., 2021) If E, G are disjoint, bounded open convex sets in $\mathbb{C}$, then there is k_0 where for $k > k_0$,

$$Z_k(E,G) \leq 16h^{-k} + \mathcal{O}(h^{-2k}).$$

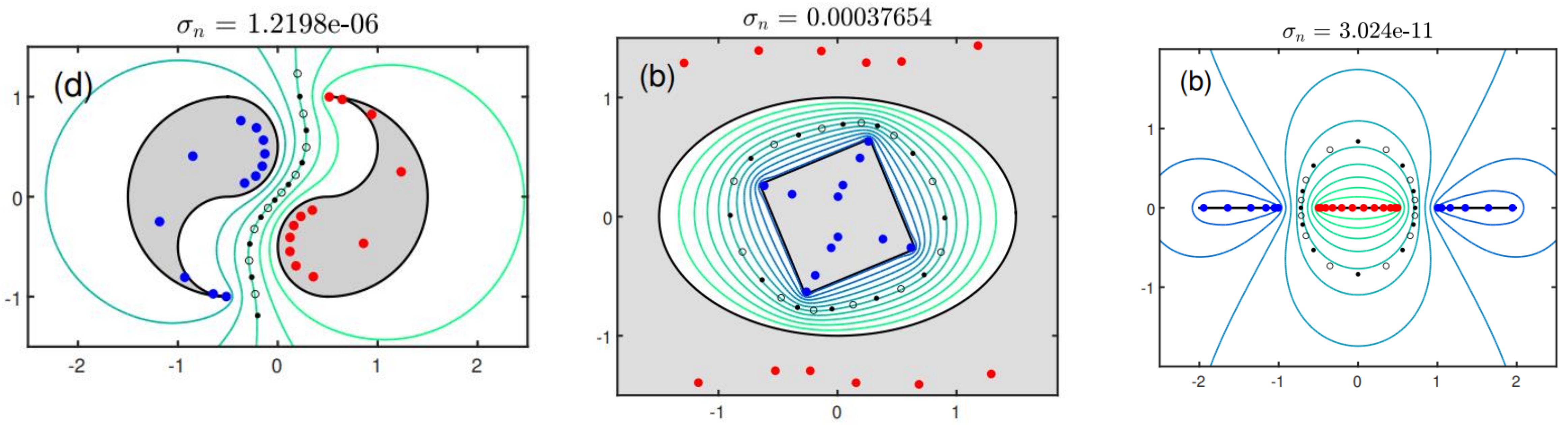
*We have an inelegant explicit upper bound and expression for k_0.





[(Gončar, 1969), (Ganelius, 1977, 1979), (Rubin, Townsend, & W., 2022)]

Computing Zolotarev rationals



Key ideas:

- Use equivalency with best sign function approximation
- New developments in AAA rational approximation algorithm for accurate approximations to sgn(z) (Trefethen, W., 2024).

[(Trefethen & W., 2024), (Nakatsukasa, Sète & Trefethen, 2018)]

The low rank connection:

Displacement, Zolotarev, and singular value decay

Matrices with displacement structure

$$A \in \mathbb{C}^{m\times m},\ B \in \mathbb{C}^{n\times n},\quad X, F \in \mathbb{C}^{m\times n},$$

$$AX - XB = F$$

"X has (A, B) displacement structure"

Appears with X as an unknown: discretization of PDEs, reduced order modeling, signal processing (see Simoncini SIAM REV, 2016 + ref. therein).

Special (A, B, F) triplets characterize properties of X for structured matrices: e.g., $X =$ Toeplitz, Hankel, Cauchy, Vandermonde, and more.

[(Beckermann & Townsend, 2019), (Kailath & Sayed, 1995) (Heinig, 1995)]

Example: Vandermonde matrix

$$DV - VQ = uv^*$$

$$\begin{pmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_m \end{pmatrix} \begin{pmatrix} (\gamma_1)^0 & (\gamma_1)^1 & \cdots & (\gamma_1)^{n-1} \\ (\gamma_2)^0 & (\gamma_2)^1 & \cdots & (\gamma_2)^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ (\gamma_m)^0 & (\gamma_m)^1 & \cdots & (\gamma_m)^{n-1} \end{pmatrix} - \begin{pmatrix} (\gamma_1)^0 & (\gamma_1)^1 & \cdots & (\gamma_1)^{n-1} \\ (\gamma_2)^0 & (\gamma_2)^1 & \cdots & (\gamma_2)^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ (\gamma_m)^0 & (\gamma_m)^1 & \cdots & (\gamma_m)^{n-1} \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} (\gamma_1)^1 & \cdots & (\gamma_1)^{n-1} & (\gamma_1)^n \\ (\gamma_2)^1 & \cdots & (\gamma_2)^{n-1} & (\gamma_2)^n \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ (\gamma_m)^1 & \cdots & (\gamma_m)^{n-1} & (\gamma_m)^n \end{pmatrix} - \begin{pmatrix} (\gamma_1)^1 & \cdots & (\gamma_1)^{n-1} & (\gamma_1)^0 \\ (\gamma_2)^1 & \cdots & (\gamma_2)^{n-1} & (\gamma_2)^0 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ (\gamma_m)^1 & \cdots & (\gamma_m)^{n-1} & (\gamma_m)^0 \end{pmatrix}$$

The low rank connection

$$A \in \mathbb{C}^{m\times m},\; B \in \mathbb{C}^{n\times n}, \quad X, F \in \mathbb{C}^{m\times n},$$

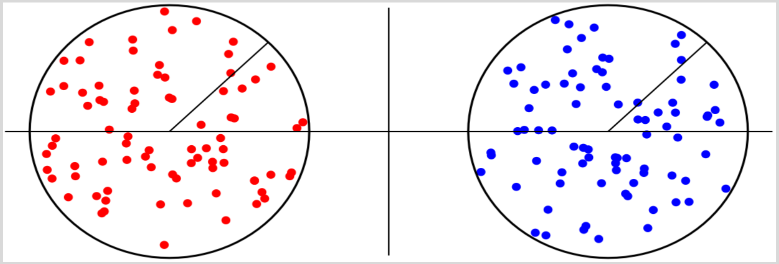
$$AX - XB = F$$

X is well-approximated by a low rank matrix when

(1) $\lambda(A)$ and $\lambda(B)$ are "well-separated"

(2) F is low rank

Example where $\lambda(A) \cap \lambda(B) = 0$



$\lambda(A) \subset E$ $\lambda(B) \subset G$

[(Beckermann & Townsend, 2019), (Sabino, 2008), (Penzl, 1999), (Benner, Truhar & Li, 2009), (Li & White, 2002), (Druskin, Knizhnerman & Simoncini, 2011), (Peaceman & Rachford, 1955), (Lu & Wachspress, 1991), (Townsend & W., 2018), (Rubin, Townsend, & W., 2022)]

The factored ADI method

$$AX - XB = F$$

<u>Factored ADI:</u> $A({\color{red}ZDY^*}) - ({\color{red}ZDY^*})B = USV^*$

U, V have ρ columns.

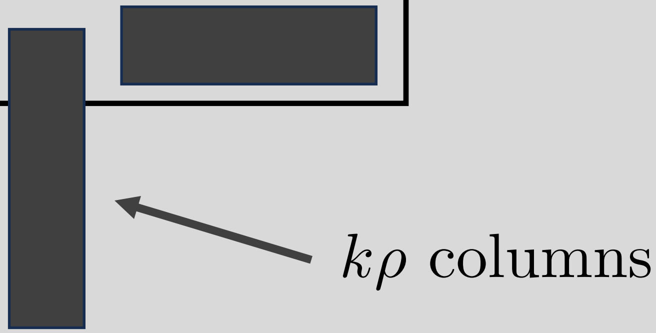
$$Z^{(k)} = \left[\ \hat{Z}^{(1)}\ \middle|\ \hat{Z}^{(2)}\ \middle|\ \cdots\ \middle|\ \hat{Z}^{(k)}\ \right], \quad \begin{cases} \hat{Z}^{(1)} = (A - \beta_1 I)^{-1} US, \\ \hat{Z}^{(i+1)} = (A - \alpha_i I)(A - \beta_{i+1} I)^{-1} Z^{(i)} \end{cases}$$

$$Y^{(k)} = \left[\ \hat{Y}^{(1)}\ \middle|\ \hat{Y}^{(2)}\ \middle|\ \cdots\ \middle|\ \hat{Y}^{(k)}\ \right], \quad \begin{cases} \hat{Y}^{(1)} = (B^* - \alpha_1 I)^{-1} V, \\ \hat{Y}^{(i+1)} = (B^* - \beta_i I)(B^* - \alpha_{i+1} I)^{-1} Y^{(i)} \end{cases}$$

$$D^{(k)} = \operatorname{diag}\left((\beta_1 - \alpha_1) I_\rho, \cdots, (\beta_k - \alpha_k) I_\rho\right)$$

$$X^{(k)} = Z^{(k)} D^{(k)} Y^{(k)^*}$$

<u>After k iterations:</u> $X^{(k)} = Z^{(k)} (W^{(k)})^*,$



[(Li & White, 2002), (Benner, Li & Truhar, 2009)]

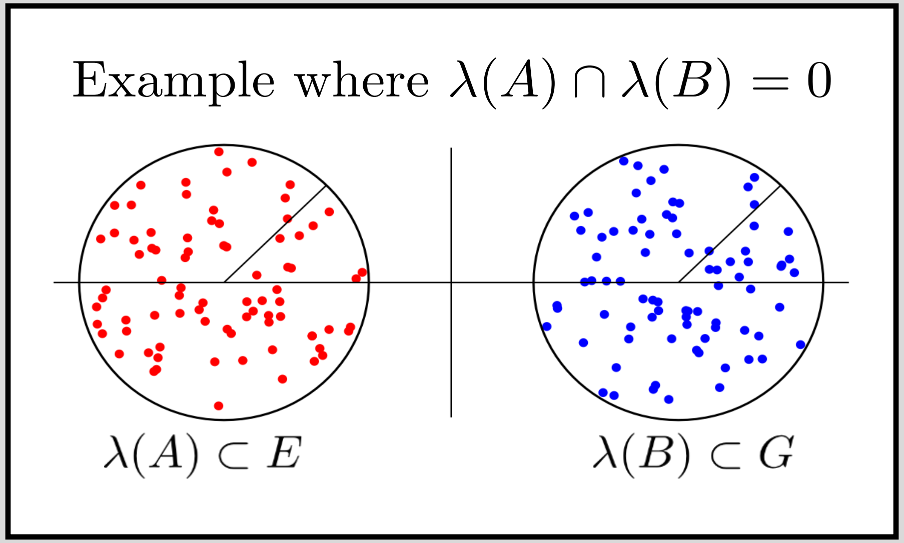
Bounding the ADI error

$$X - X^{(k)} = r_k(A) X r_k(B)^{-1}, \quad r_k(z) = \prod_{j=1}^{k} \frac{z-\alpha_j}{z-\beta_j}$$

$$\|X - X^{(k)}\|_2 \le \|r_k(A) r_k(B)^{-1}\|_2 \|X\|_2 \le \|r_k(\lambda(A))\|_2 \|r_k(\lambda(B))^{-1}\|_2 \|X\|_2$$

$$\|X - X^{(k)}\|_2 \le \frac{\max_{z\in E} |r(z)|}{\min_{z \in G} |r(z)|} \|X\|_2$$

$$\sigma_{\rho k+1} \le \|X - X^{(k)}\|_2 \le Z_k(E,G) \|X\|_2$$



[(Beckermann & Townsend, 2017), (Sabino, 2008), (Penzl, 1999), (Benner, Truhar & Li, 2009), (Li & White, 2002), (Druskin, Knizhnerman & Simoncini, 2011), (Peaceman & Rachford, 1955), (Lu & Wachspress, 1991), (Townsend & W., 2018), (Rubin, Townsend, & W., 2022)]

Bounding the Zolotarev numbers

$$AX - XB = F$$

Theorem (informal):

If $\text{rank}(F) \leq \rho$, and $\lambda(A)$, $\lambda(B)$ are each arcs on the unit circle disjoint from one-another, then there are Z, W, each with ρk columns such that

$$\|ZW^* - X\|_2 / \|X\|_2 \leq 4 \exp\left(\frac{\pi^2}{2 \log 16\gamma}\right)^{-\lfloor k/\rho \rfloor},$$

where γ is the cross-product of the arcs.

Key Idea: Use fADI to construct $ZW^* \approx X$

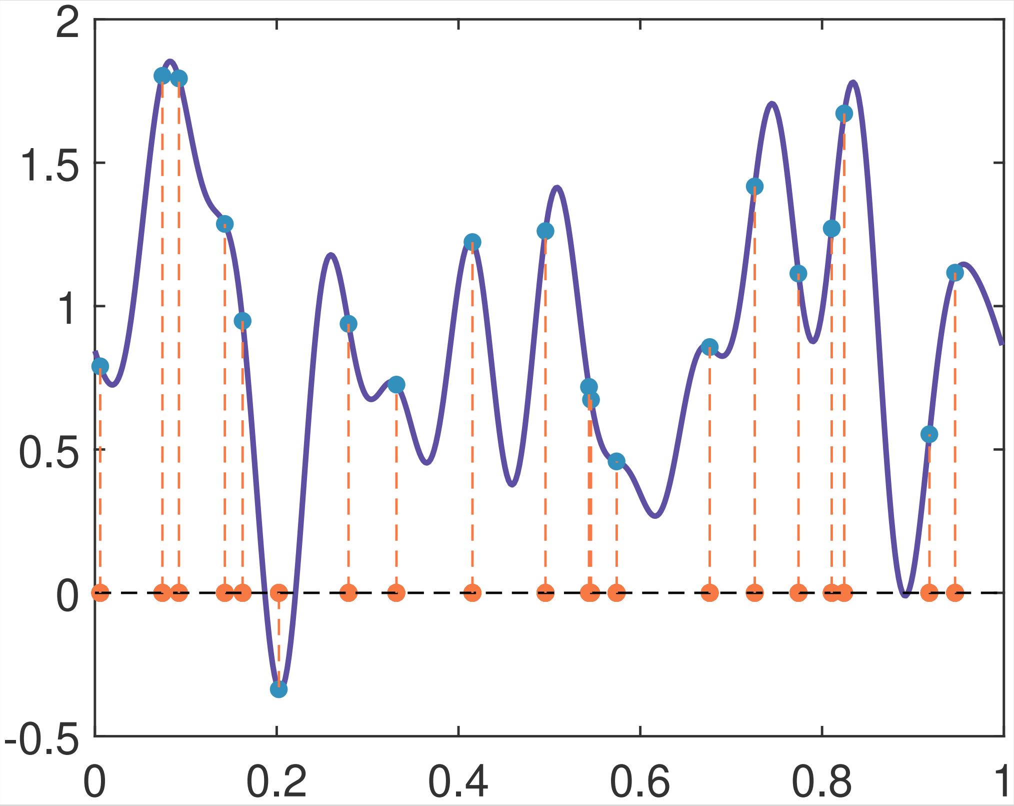
[(W., 2021), (W., Epperly, & Barnett, 2024)]

The inverse nonuniform discrete Fourier transform

Inverse nonuniform discrete Fourier transforms

$$b_j = \sum_{k=0}^{n-1} c_k e^{-2\pi i p_j (k-1)}, \quad 0 \leq j \leq m-1, \quad p_j \in [0,1]$$

Goal: Recover coefficients $c_0, \ldots, c_{n-1}$.



[Forward problem: (Barnett, see FINUFFT package), (Potts, Kunis, see NFFT package) (Greengard and Inati, 2006), (Townsend and Ruiz, 2017), Other direct inversion solvers: (Kircheis and Potts, 2019), (Dutt and Rokhlin, 1993)]

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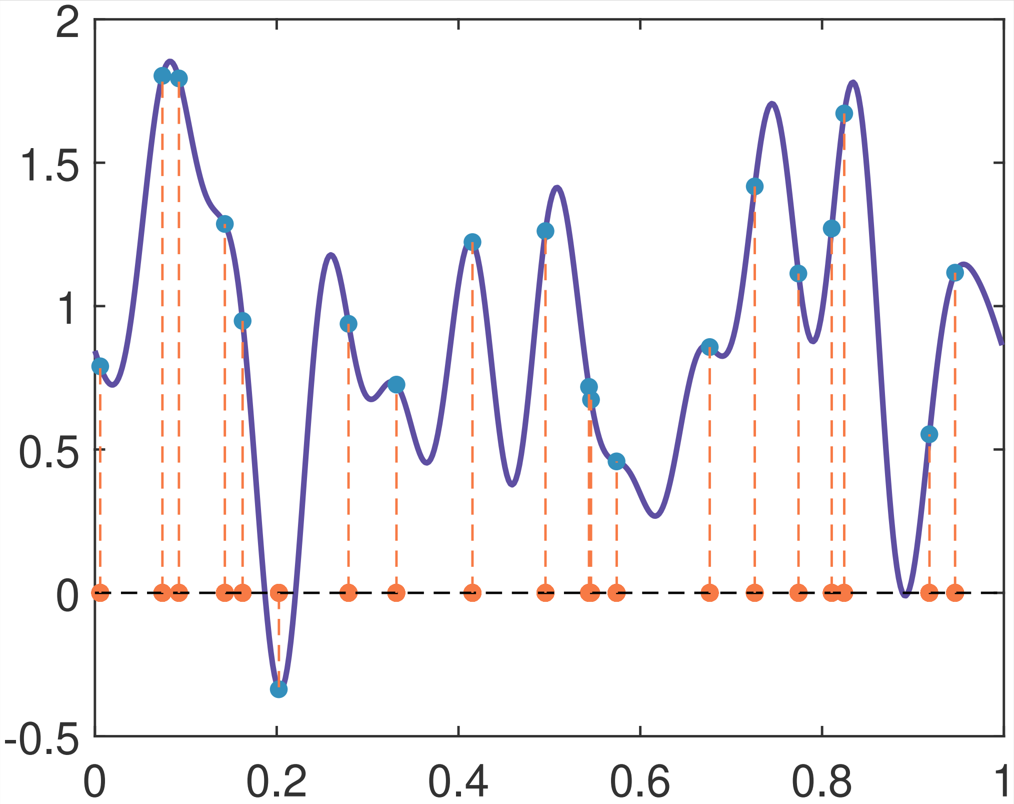

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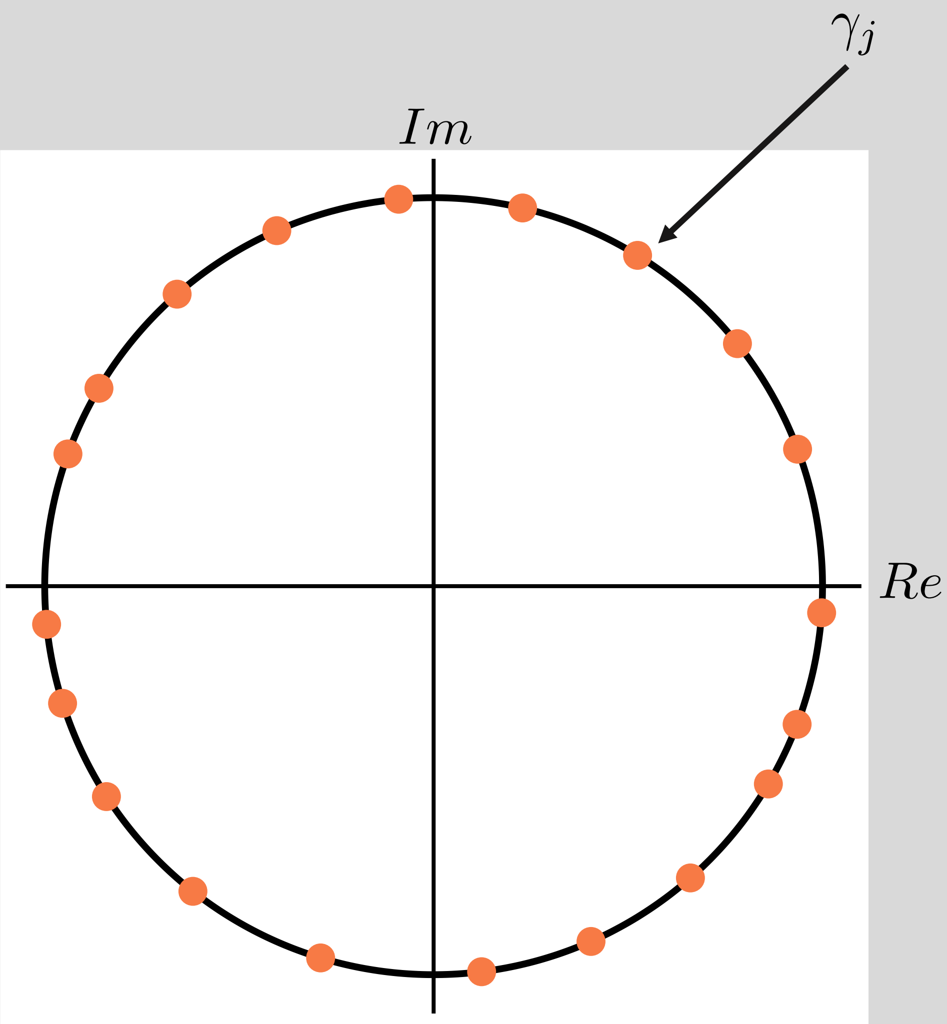
Goal: Recover coefficients $c_0, \ldots, c_{n-1}$.

Set $\gamma_j = e^{-2\pi i p_j}$

$$\begin{bmatrix} (\gamma_1)^0 & (\gamma_1)^1 & \cdots & (\gamma_1)^{n-1} \\ (\gamma_2)^0 & (\gamma_2)^1 & \cdots & (\gamma_2)^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ (\gamma_m)^0 & (\gamma_m)^1 & \cdots & (\gamma_m)^{n-1} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}$$

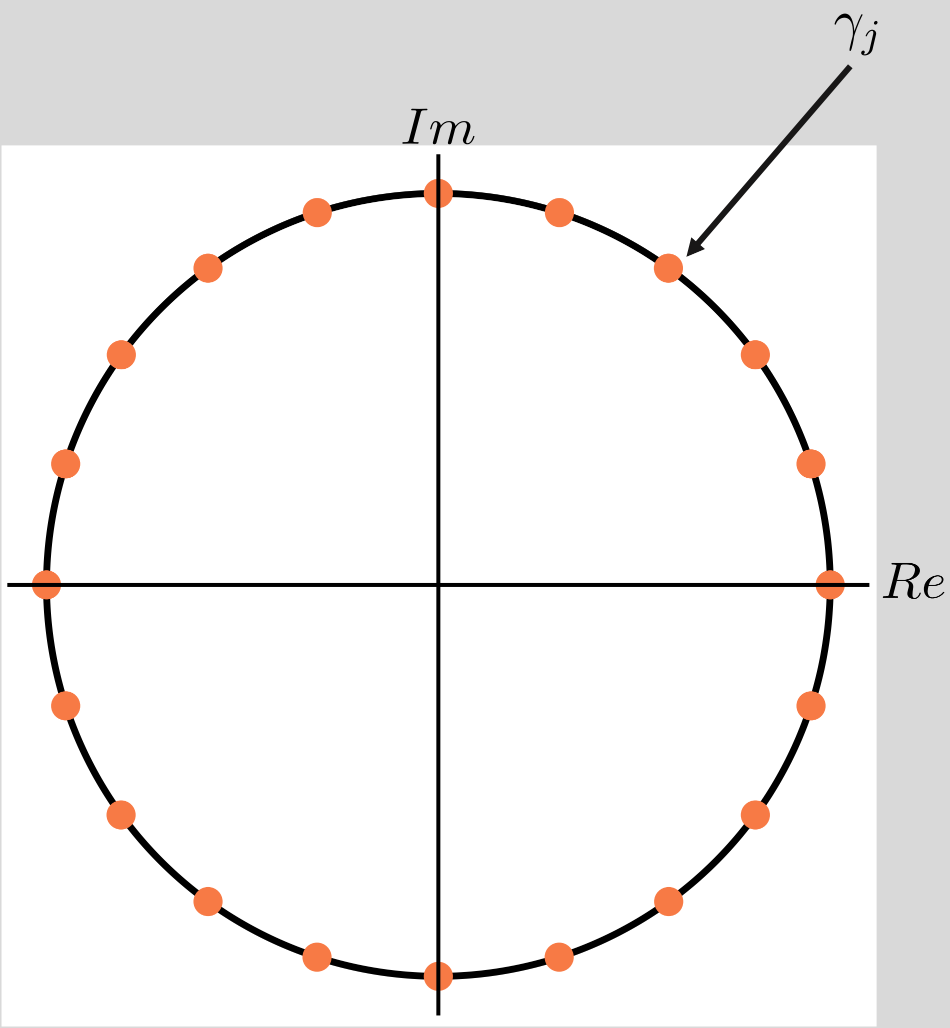
Solve a (overdetermined) linear system $Vc = b$



Inverse nonuniform discrete Fourier transforms

Goal: Solve a linear system $Vc = b$

When $\gamma_1, \ldots, \gamma_m$ are equally spaced on the unit circle and $m = n$...



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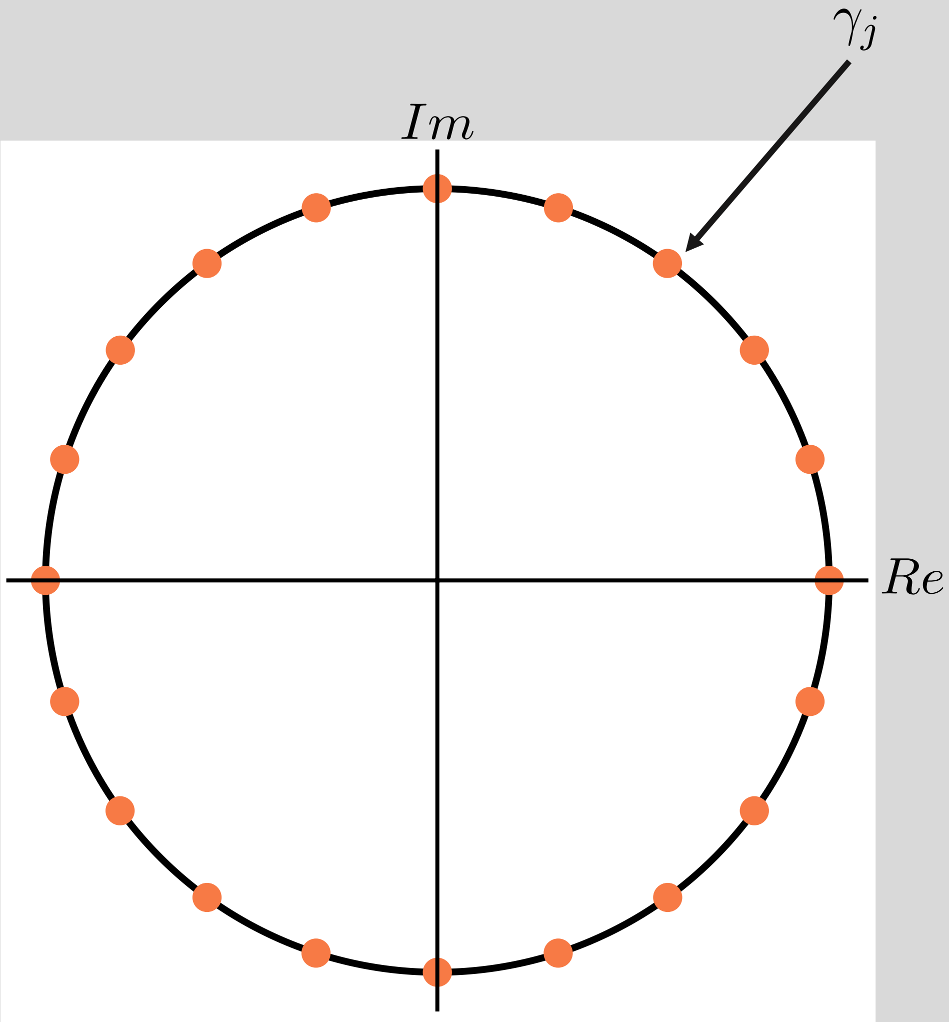
$$V^{-1} = V^*,\ c = V^* b$$

Compute in $\mathcal{O}(n \log n)$ flops via FFT



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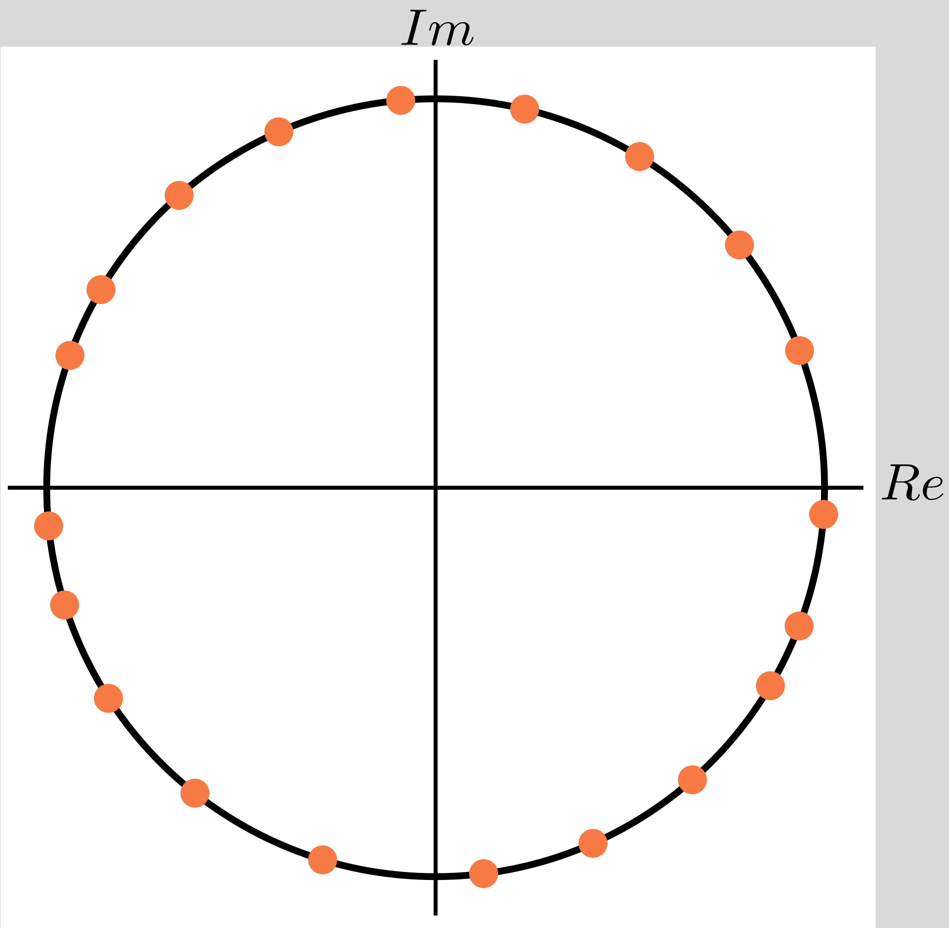


Small perturbation, structure breaks!

Solution: Find a new structure

Inverse nonuniform discrete Fourier transforms

V^*V is Toeplitz

Iterative approach:

(1) Form $V^*Vc = V^*b$.

(2) Apply iterative method with fast matrix-vector multiply

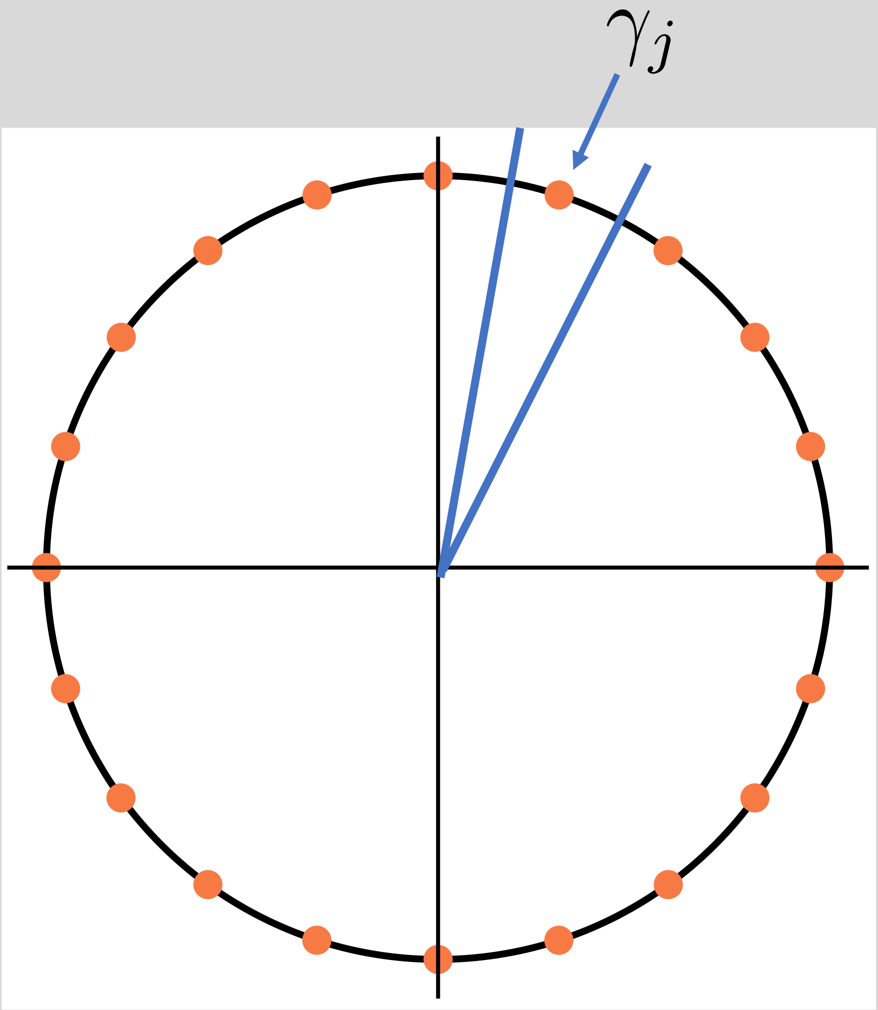
Problem:

Depends on $\kappa(V^*V) = \|V^*V\|_2\|(V^*V)^{-1}\|_2$!

Solution: Fast direct solvers

Solve cost does not depend on $\kappa(V)$

Also good for multiple right-hand sides!



Inverse nonuniform discrete Fourier transforms

When the problem compels it, use a direct solver!

Our wishlist

(1) (Super)fast!

(2) Do not square condition number

(3) Can handle overdetermined case

(4) "Black box"

```
x = inufft(samplelocs, n, rhs, acc)
```

[Related ideas: (Greengard and Inati, 2006), (Kircheis and Potts, 2019),(Dutt and Rokhlin, 1993), (Heinig, 1995)]

The fast Cauchy-like transformation family

$$AX - XB = F, \text{ with } F \text{ low rank}$$

$$+$$

$$A, B \text{ are diagonal or ``very easily diagonalizable''}.$$

The Cauchy-like transformation

$$V \to C = VF^*$$

$$DV - VQ = uw^*$$

$$D(VF^*) - (VF^*)FQF^* = uw^*F^*$$

$$FQF^* = \Lambda = \mathrm{diag}(\omega^2, \omega^4, \cdots, \omega^{2n}), \quad \omega = e^{i\pi/n}.$$

$$\Longrightarrow \quad DC - C\Lambda = u\tilde{w}^*$$

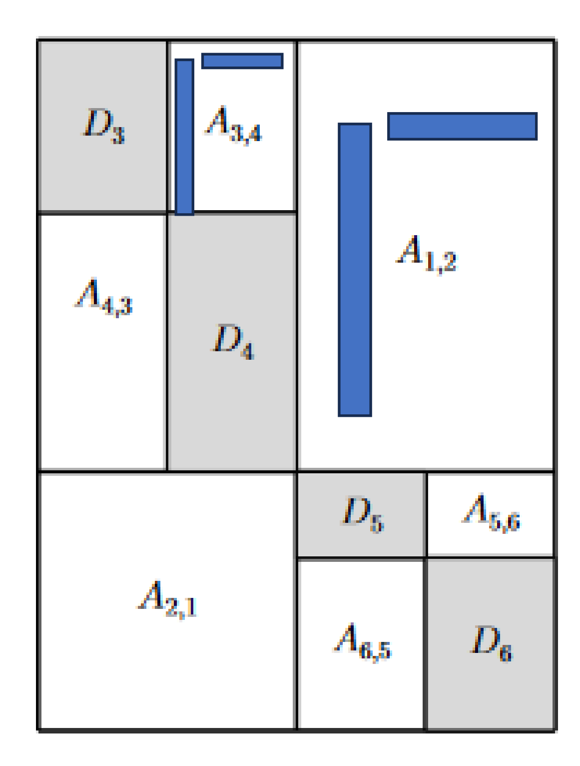
Cauchy-like = "diagonal" displacement structure with low rank RHS.

$$\Longrightarrow \quad C_{jk} = \frac{(u\tilde{w}^*)_{jk}}{D_{jj} - \Lambda_{kk}}$$

If $V \in \mathbb{C}^{m\times n}$, only 1 length-n FFT to (implicitly represent C.

The Cauchy-like transformation

C has hierarchical low rank structure (C is an HSS matrix)



The Cauchy-like transformation

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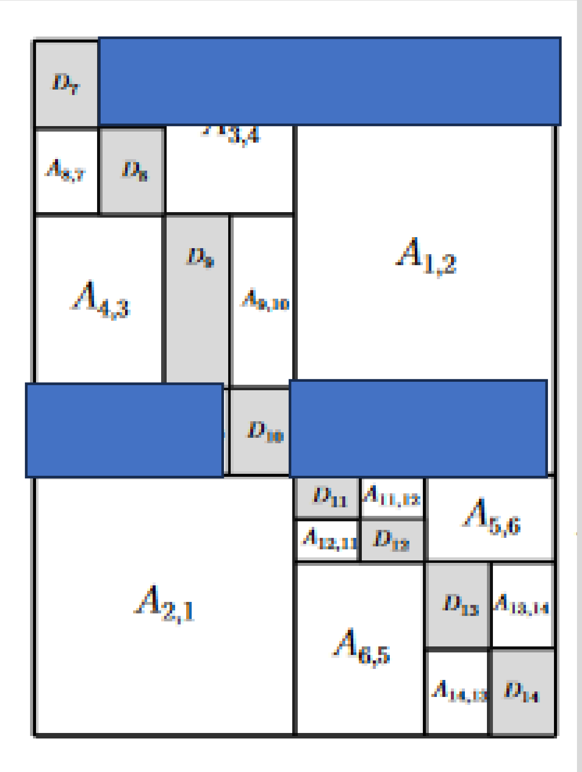
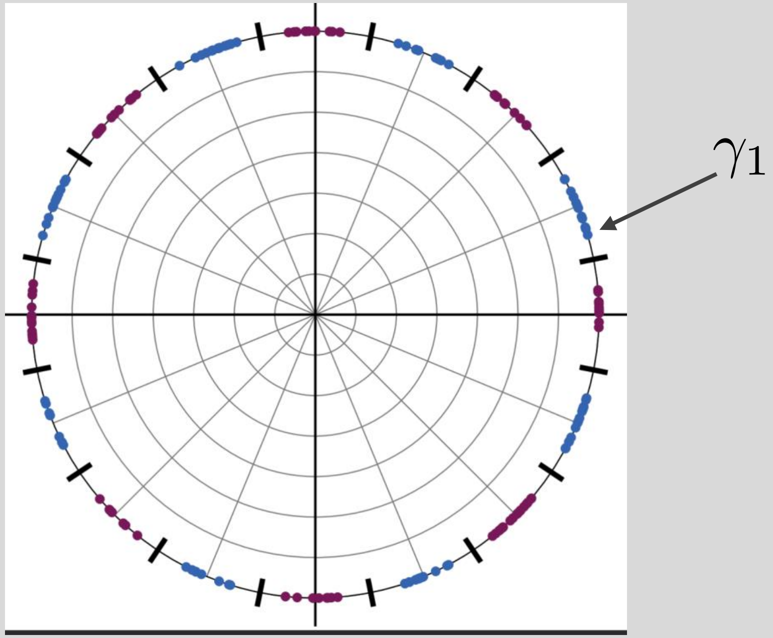
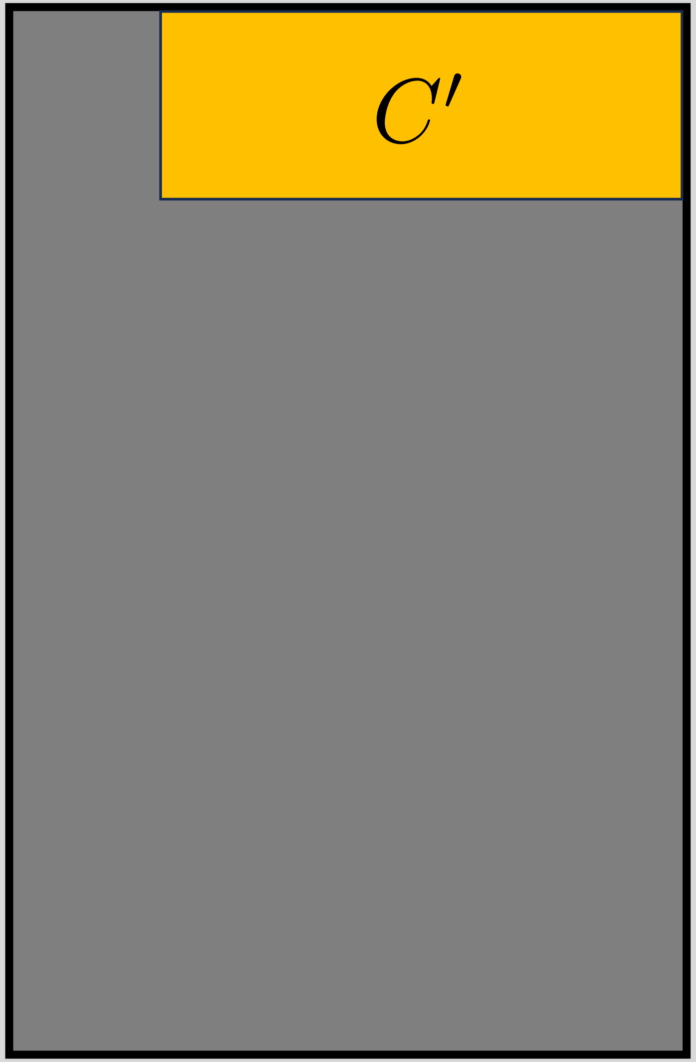


Low rank submatrices

$$DC - C\Lambda = u(Fv)^*$$

$$D = \begin{bmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_m \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \omega^2 & & & \\ & \omega^4 & & \\ & & \ddots & \\ & & & \omega^{2n} \end{bmatrix}$$





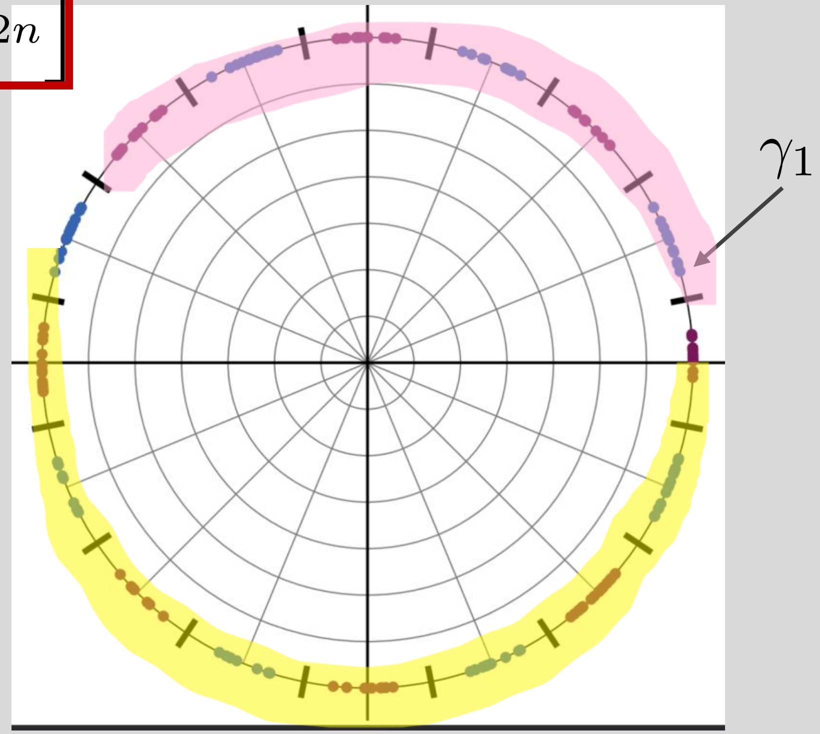
Low rank submatrices

$$DC - C\Lambda = u(Fv)^*$$

$$D = \begin{bmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_m \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \omega^2 & & & \\ & \omega^4 & & \\ & & \ddots & \\ & & & \omega^{2n} \end{bmatrix}$$

For the submatrix C' of C

$$\begin{bmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_j \end{bmatrix} C' - C' \begin{bmatrix} \omega^\ell & & \\ & \ddots & \\ & & \omega^{2n} \end{bmatrix} = [u(Fv)^*]_{JL}$$



ADI-based hierarchical compression

$$D_J C' - C'\Lambda_L = [u(Fv)^*]_{JL}$$

$$\|C' - ZW^*\|_2 \leq \epsilon \|C'\|_2,$$

Z, W have $k = \mathcal{O}(\log n \log 1/\epsilon)$ columns.

We construct Z, W via one-sided ADI-based interpolative decomposition.

$$Z^{(k)} = \left[\ \hat{Z}^{(1)} \mid \hat{Z}^{(2)} \mid \cdots \mid \hat{Z}^{(k)}\ \right], \quad \begin{cases} \hat{Z}^{(1)} = (A - \beta_1 I)^{-1} US, \\ \hat{Z}^{(i+1)} = (A - \alpha_i I)(A - \beta_{i+1} I)^{-1} Z^{(i)} \end{cases}$$

$$Y^{(k)} = \left[\ \hat{Y}^{(1)} \mid \hat{Y}^{(2)} \mid \cdots \mid \hat{Y}^{(k)}\ \right], \quad \begin{cases} \hat{Y}^{(1)} = (B^* - \alpha_1 I)^{-1} V, \\ \hat{Y}^{(i+1)} = (B^* - \beta_i I)(B^* - \alpha_{i+1} I)^{-1} Y^{(i)} \end{cases}$$

$$D^{(k)} = \operatorname{diag}\left((\beta_1 - \alpha_1) I_\rho, \cdots, (\beta_k - \alpha_k) I_\rho\right)$$

$$X^{(k)} = Z^{(k)} D^{(k)} {Y^{(k)}}^*$$

ADI-based hierarchical compression

$$D_J C' - C' \Lambda_L = [u(Fv)^*]_{JL}$$

$$\|C' - ZW^*\|_2 \leq \epsilon \|C'\|_2,$$

Z, W have $k = \mathcal{O}(\log n \log 1/\epsilon)$ columns.

We construct Z, W via one-sided ADI-based interpolative decomposition.

Total cost for low rank compression: $\mathcal{O}((n+m)\log^2 n \log^2(1/\epsilon))$

ADI-based hierarchical solver

Algorithm 1 A superfast least squares solver for $Vc = b$. (Type-II NUDFT inversion) (Transforms to $Cy = b$, solves, transforms back)

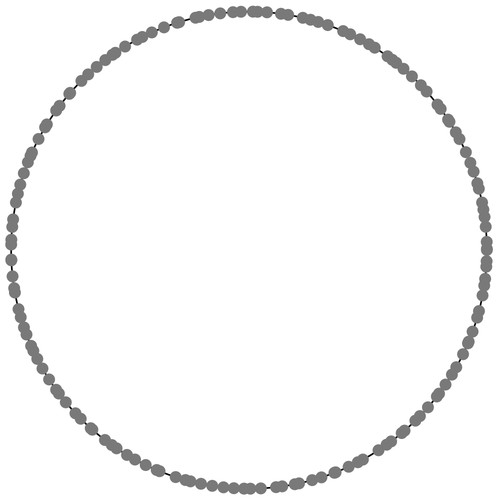
1: Compute $w = Fv$.
2: Use ADI on $DC - C\Lambda = uw^*$ to generate H, an approximate hierarchical factorization of C.
3: Solve $Hy = b$ in the least squares sense
4: Compute $c = F^*y$.

- $\mathcal{O}((m+n)\log^2(n)\log^2(1/\epsilon))$ flops, where ϵ is an accuracy parameter.
- Construction of H is automatic.
- Toeplitz version for normal equations.
- Especially effective for multiple RHSs

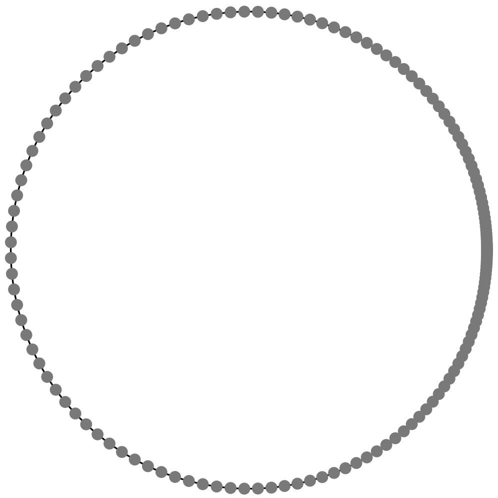
[A least-squares Toeplitz solver: (Xi, Xia, Cauley, Balakrishnan, 2014)]

When do iterative solvers work well?

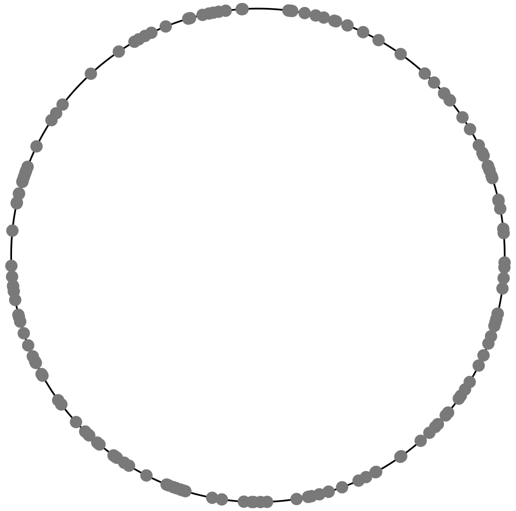
1. Jittered grid (almost equispaced)



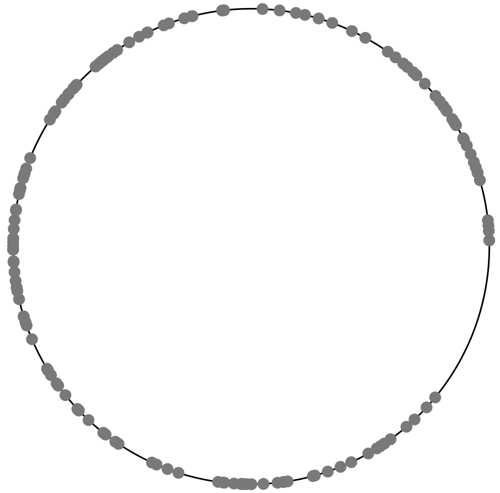
2. Clenshaw-Curtis quadrature nodes



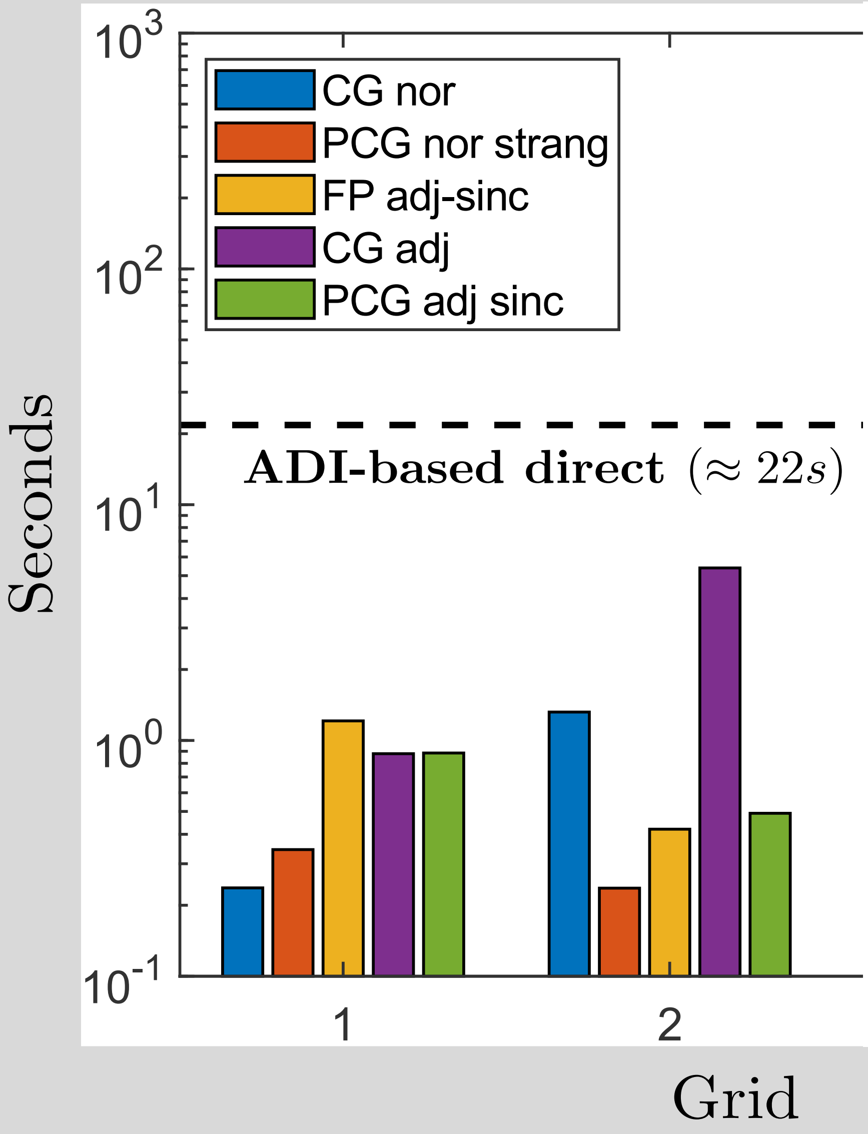
3. Random iid from uniform distribution



4. Random + gaps



When iterative methods work well



Normal equations $(V^*Vx = V^*b)$

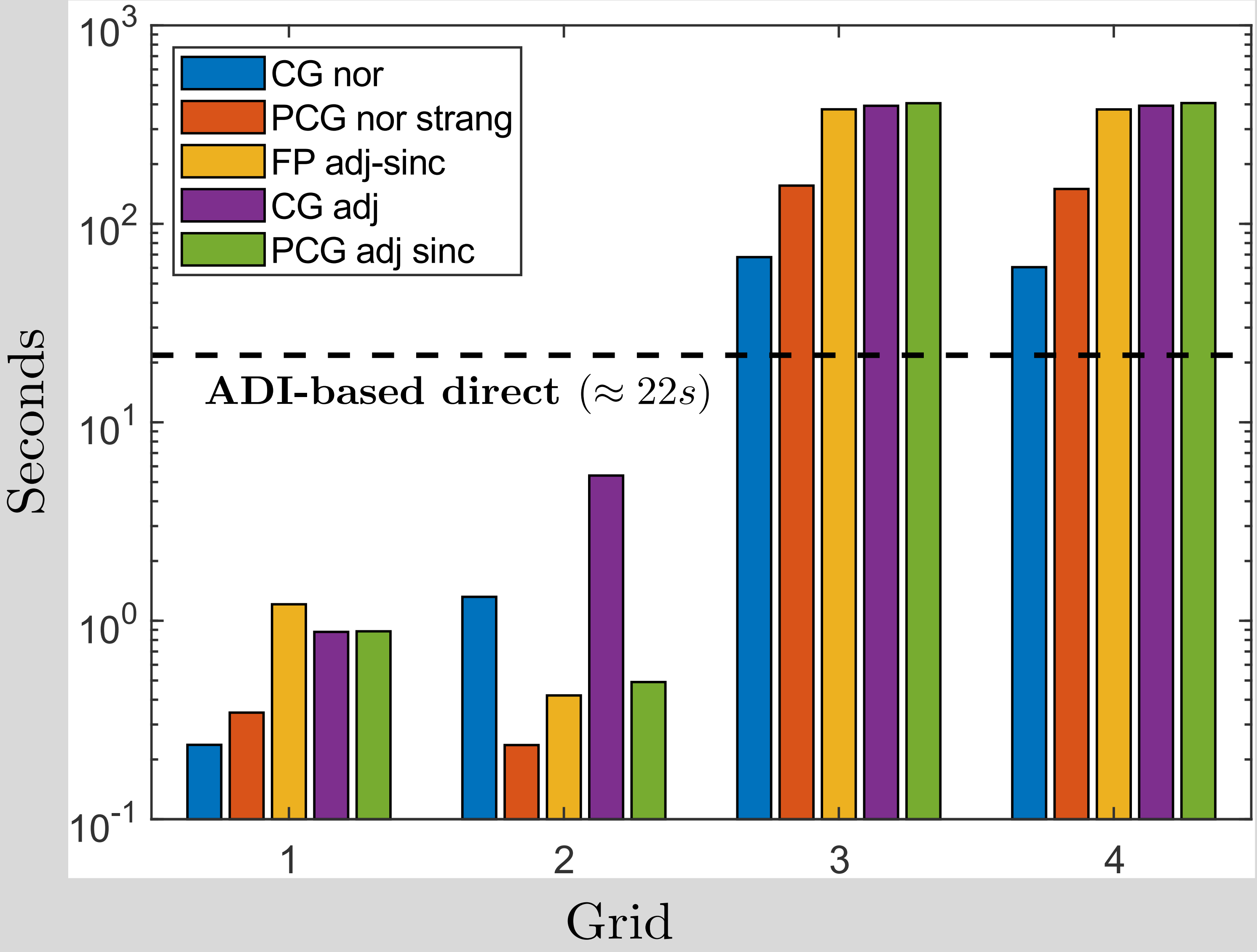
- CG
- CG w/Strang precond.

Adjoint normal equations $(VV^*y = b)$

- Greengard-Inati fixed point
- CG
- CG with sinc-quad. precond.

$m = 524,288$ and $n = m/2$

When a direct solver is needed



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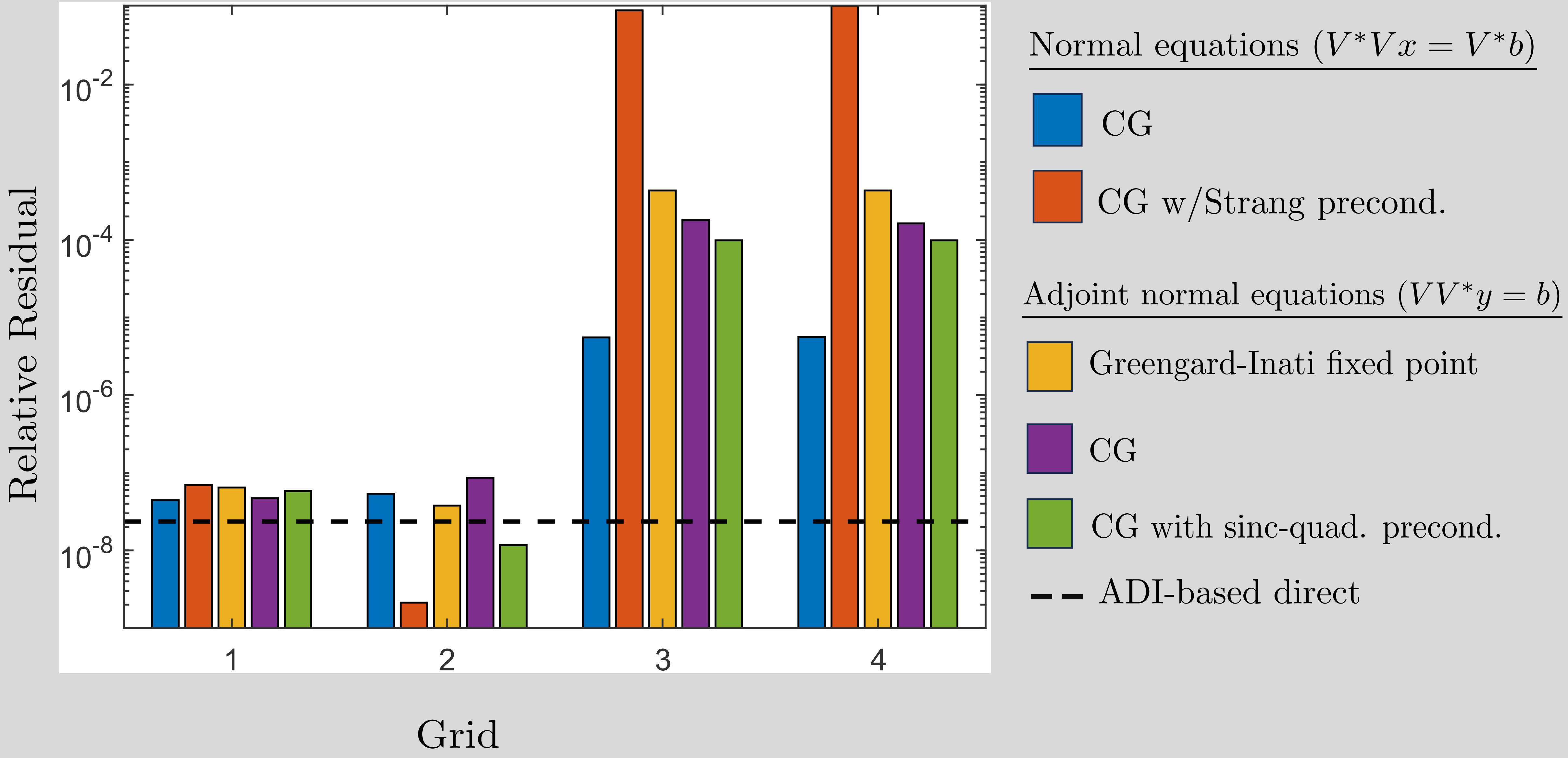
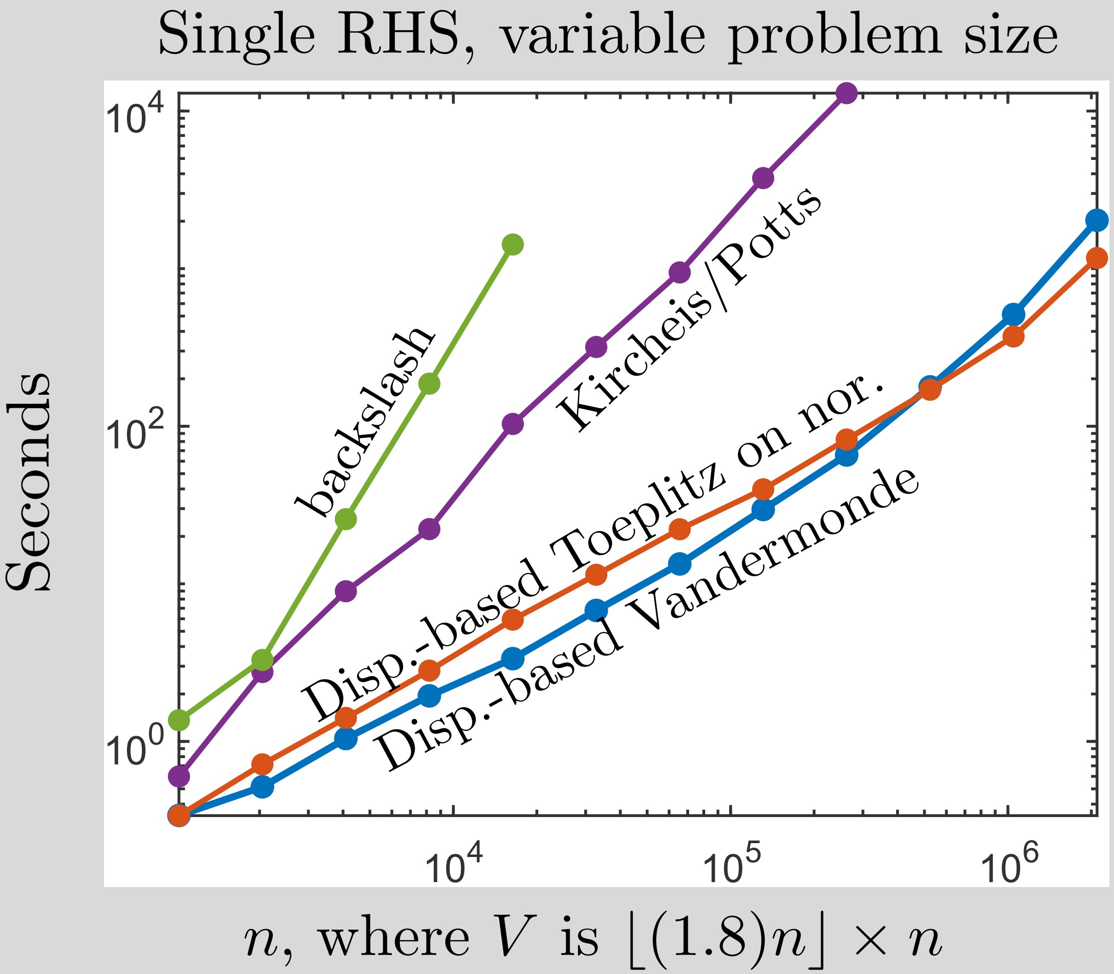
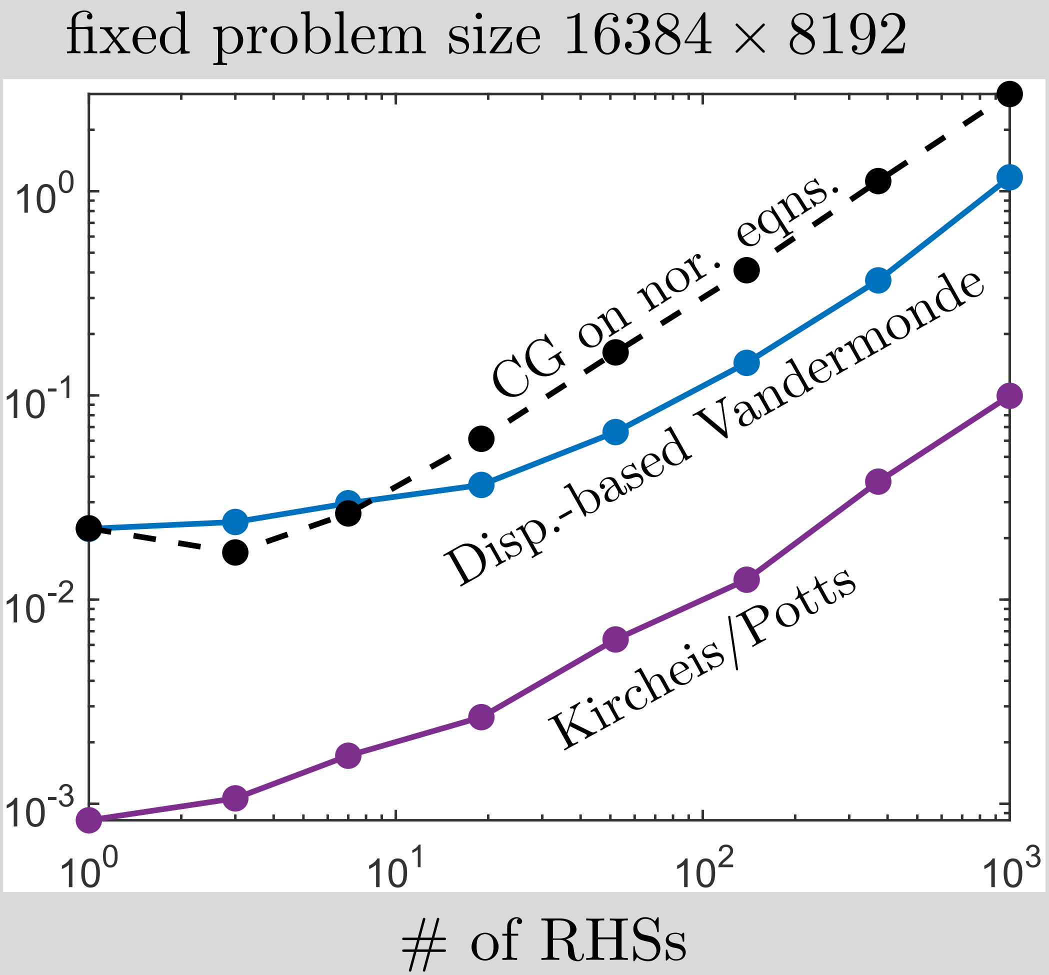


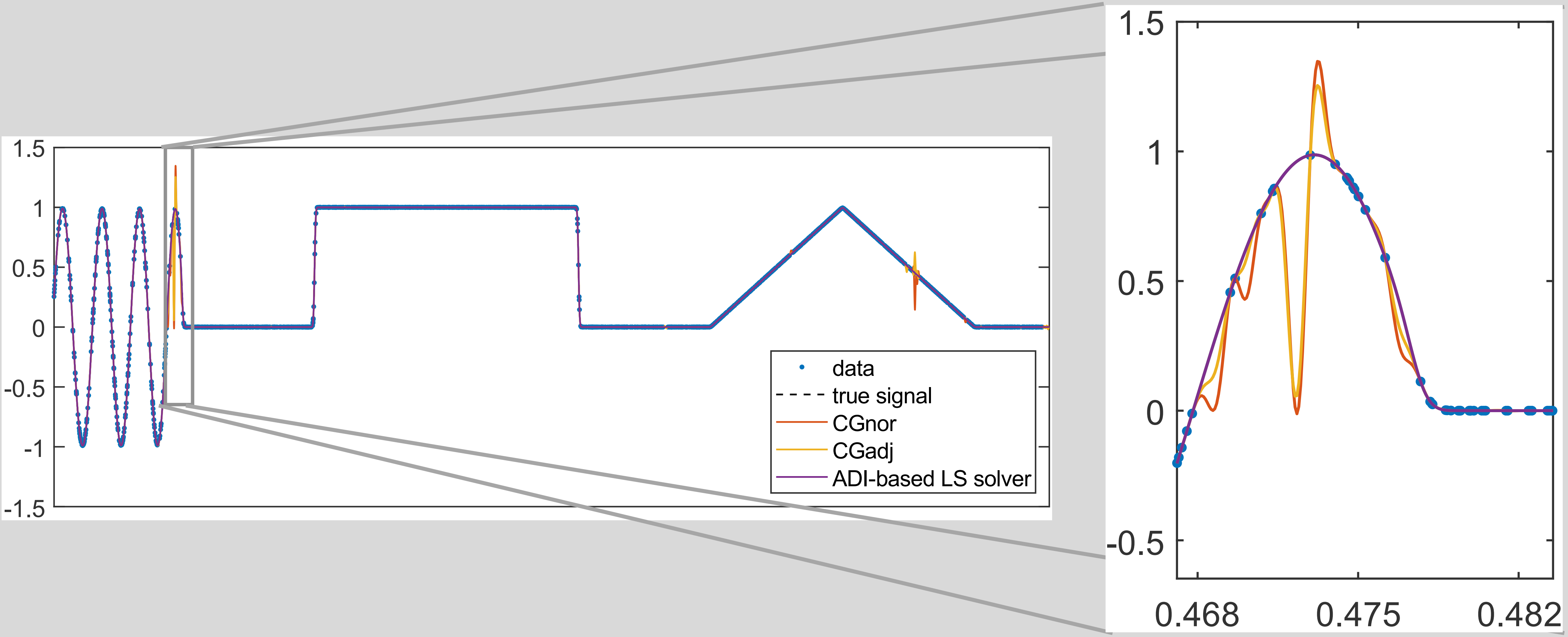
Systems with many right-hand sides





[(Kircheis and Potts, 2019)]

Solution properties



1D signal reconstructed using samples from Grid 3

Ongoing work

The 2D NUDFT

$$V_{2D} = V_\epsilon * V_\eta, \; * = \text{the ``face-splitting'' product.}$$

Requires working with block-structured matrices

Blocks of Cauchy-like matrices, linked compression properties across blocks.
$\rightarrow$ one large HSS matrix.

Several related block-structured matrices in other applications
(e.g., block–Toeplitz with Toeplitz blocks)

Noise and related issues

Basic Tikhonov regularization is straightforward.

Various constrained optimization problems with rank-structured matrices?

Designing preconditioners, fast matvecs, etc.

Ongoing work

Zolotarev rationals on general sets via interpolation (with L.N. Trefethen)

Inverse-free iterative solvers for Sylvester matrix equations via Akheizer polynomials (with T. Trogdon and C. Ballew)

Resources and software

Paper on NUDFT inversion: https://arxiv.org/abs/2404.13223

Code for solver: https://github.com/heatherw3521/NUDFT

Paper on computing Zolotarev rationals: https://arxiv.org/abs/2408.14092

heatherw3521.github.io

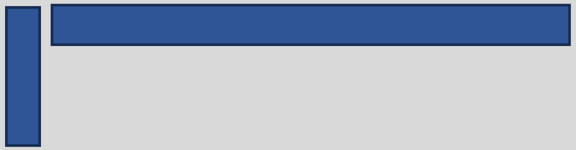
Begin Extra Slides

Constructing low rank approximations

$$D_J C' - C'\Lambda_L = [u(Fv)^*]_{JL}$$

$$C' \approx$$

$$ZW^* =$$



$$\|ZW^* - C'\|_2 \leq 4\xi^{-k}\|C'\|_2.$$

The factored ADI method:

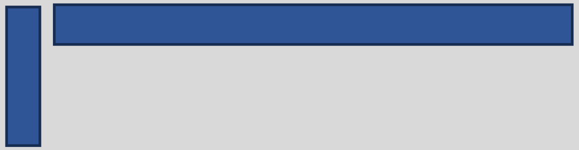
If C' is $p \times q$, we can construct:

Z in $\mathcal{O}(pk)$ operations

W in $\mathcal{O}(qk)$ operations

An ADI-based interpolative decomposition:

$$C' \approx Z'C(\mathcal{I}_P, :)$$



Only requires $\mathcal{O}(pk^2)$ operations

[Factored ADI: (Benner, Li, Truhar, 2009), Interpolative decomposition: (Cheng, Gimbutas, Martinsson, Rokhlin, 2005)]

The ADI method

<u>One ADI iteration:</u>

1. Solve $(A - \beta_{j+1} I) X^{(j+1/2)} = X^{(j)}(B - \beta_{j+1} I) + F$ for $X^{(j+1/2)}$.
2. Solve $X^{(j+1)}(B - \alpha_{j+1} I) = (A - \alpha_{j+1} I) X^{(j+1/2)} - F$ for $X^{(j+1)}$.

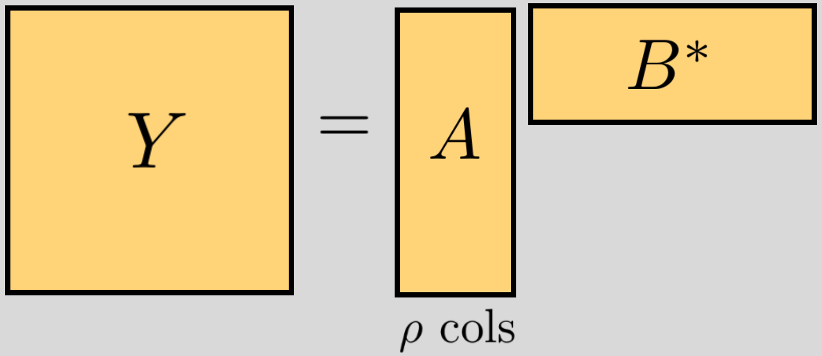
- Developed as a method for solving the heat equation by applying a splitting scheme to Crank-Nicholson.
  - Running ADI on the heat equation at steady state leads to an ADI-based Poisson solver.
    - Systems and control theory groups + numerical linear algebra groups generalized and applied the Poisson solver to develop low rank solvers for Lyapunov and Sylvester matrix equations.

- Studied by optimization community as a special instance of the alternating direction method of multipliers (ADMM).

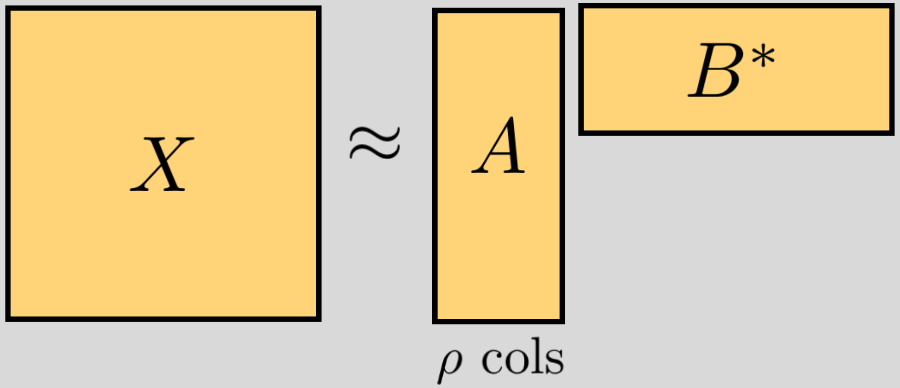
[(Peaceman & Rachford, 1955), (Lebedev, 1977), (Lions & Mercier, 1979), (Lu & Wachspress, 1991)]

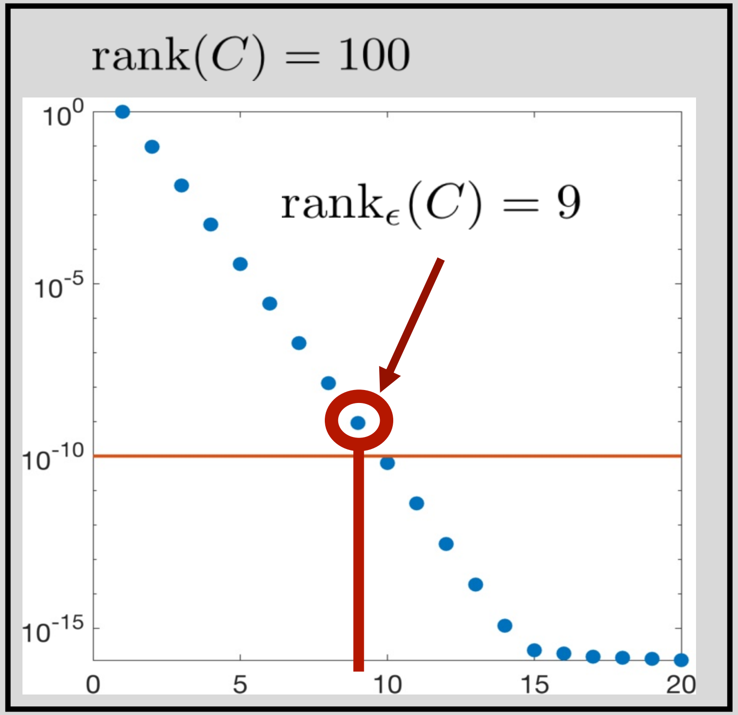
Low rank structures

$$\mathrm{rank}(Y) \leq \rho$$



$$\mathrm{rank}_\epsilon(X) \leq \rho$$





$$\sigma_{\rho+1}(X) = \min\{\|X - Y\|_2,\ \mathrm{rank}(Y) = \rho\}$$

$$\mathrm{rank}_\epsilon(X) = \text{smallest } \rho \text{ where } \sigma_{\rho+1}(X) \leq \varepsilon\|X\|_2$$