# Compression properties in rank-structured solvers for Toeplitz linear systems


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# Toeplitz linear systems

$$T = \begin{pmatrix} t_0 & t_{-1} & \cdots & \cdots & t_{-(n-1)} \\ t_1 & t_0 & \ddots & \ddots & t_{-(n-2)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & t_{-1} \\ t_{n-1} & \cdots & \cdots & t_1 & t_0 \end{pmatrix}$$

$$Tx = b$$

- Signal/Image processing
- Time series analysis
- Numerical PDEs and integral eqns.
- Large scale dynamical systems

<u>Fast</u>

$\mathcal{O}(n^2)$

*Levinson Algorithm* [Levinson (1947), Durbin (1962)]

*Schur Algorithm* [Schur (1917), Lev-Ari (1983), Kailath and Sayed (1991)]

*Displacement-based GE* [Heinig (1995), Gohberg, Kailath, and Olshevsky (1995)]

<u>Superfast</u>

$\mathcal{O}(n \log^k n)$

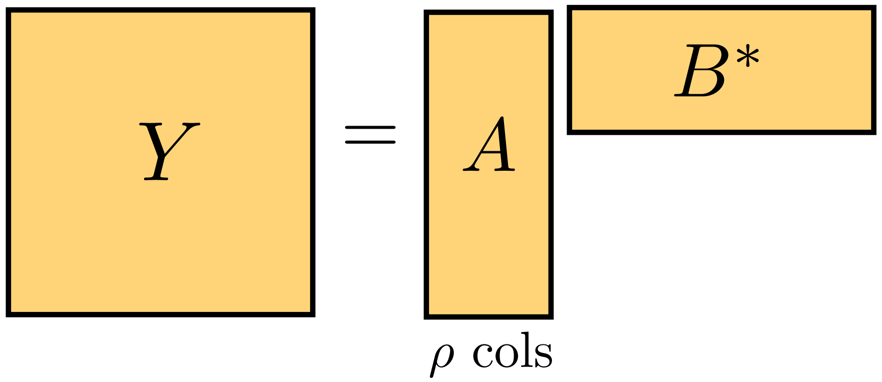
$(k \leq 4)$

**A Toeplitz matrix is just a fast transform away from being a rank-structured (compressible) matrix.**

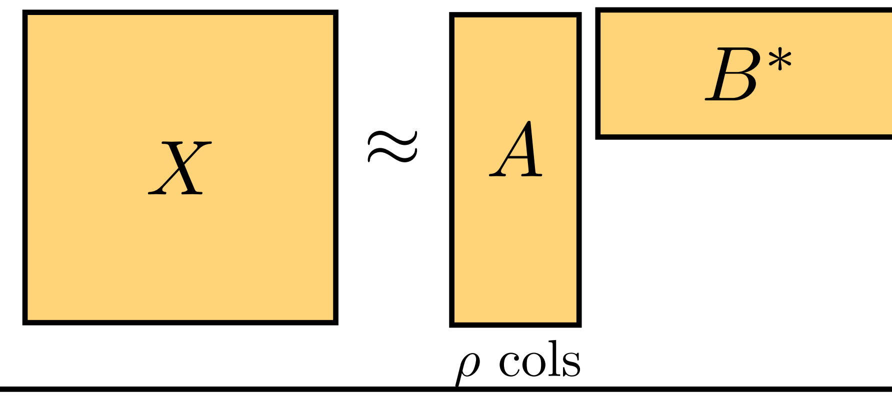
[ Martinsson, Rokhlin, Tygert (2005), Chandrasekaran, Gu, Xia, Zhu (2007), Xia, Xi, Gu (2012)]

# Low rank approximation

$$\operatorname{rank}(Y) \le \rho$$

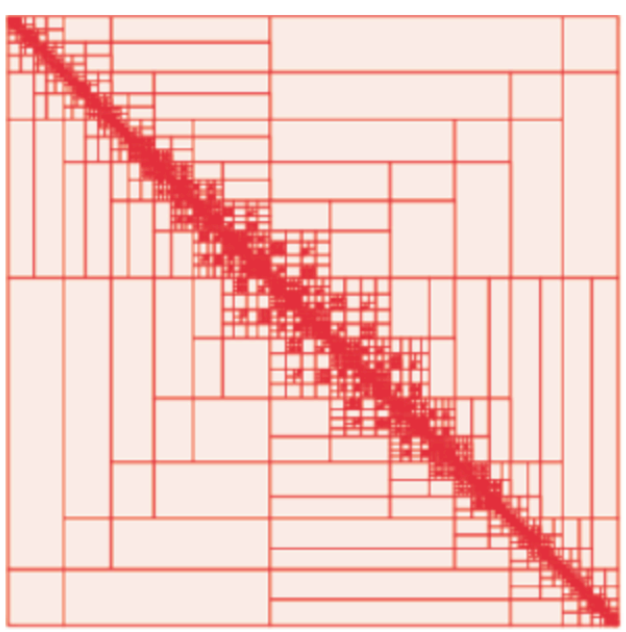



$$\operatorname{rank}_\epsilon(X) \le \rho$$

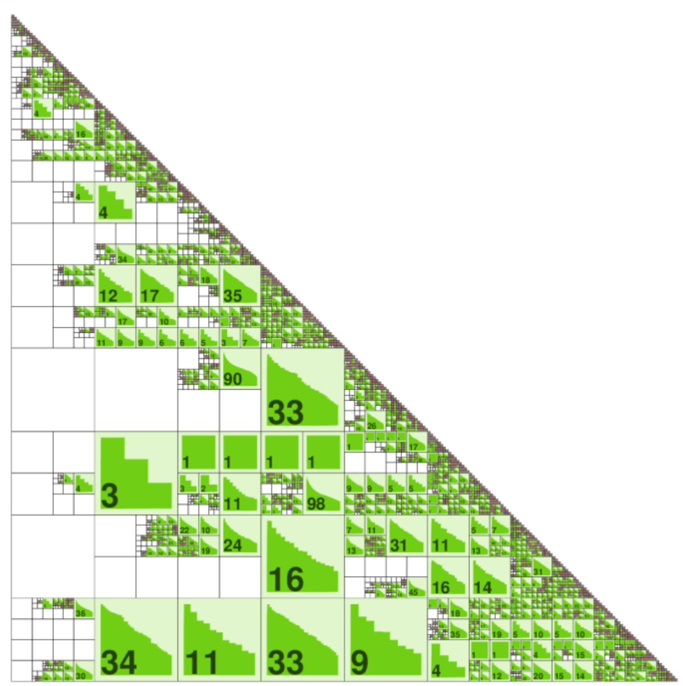



$$\sigma_{\rho+1}(X) = \min\{\|X - Y\|_2,\ \operatorname{rank}(Y) = \rho\}$$

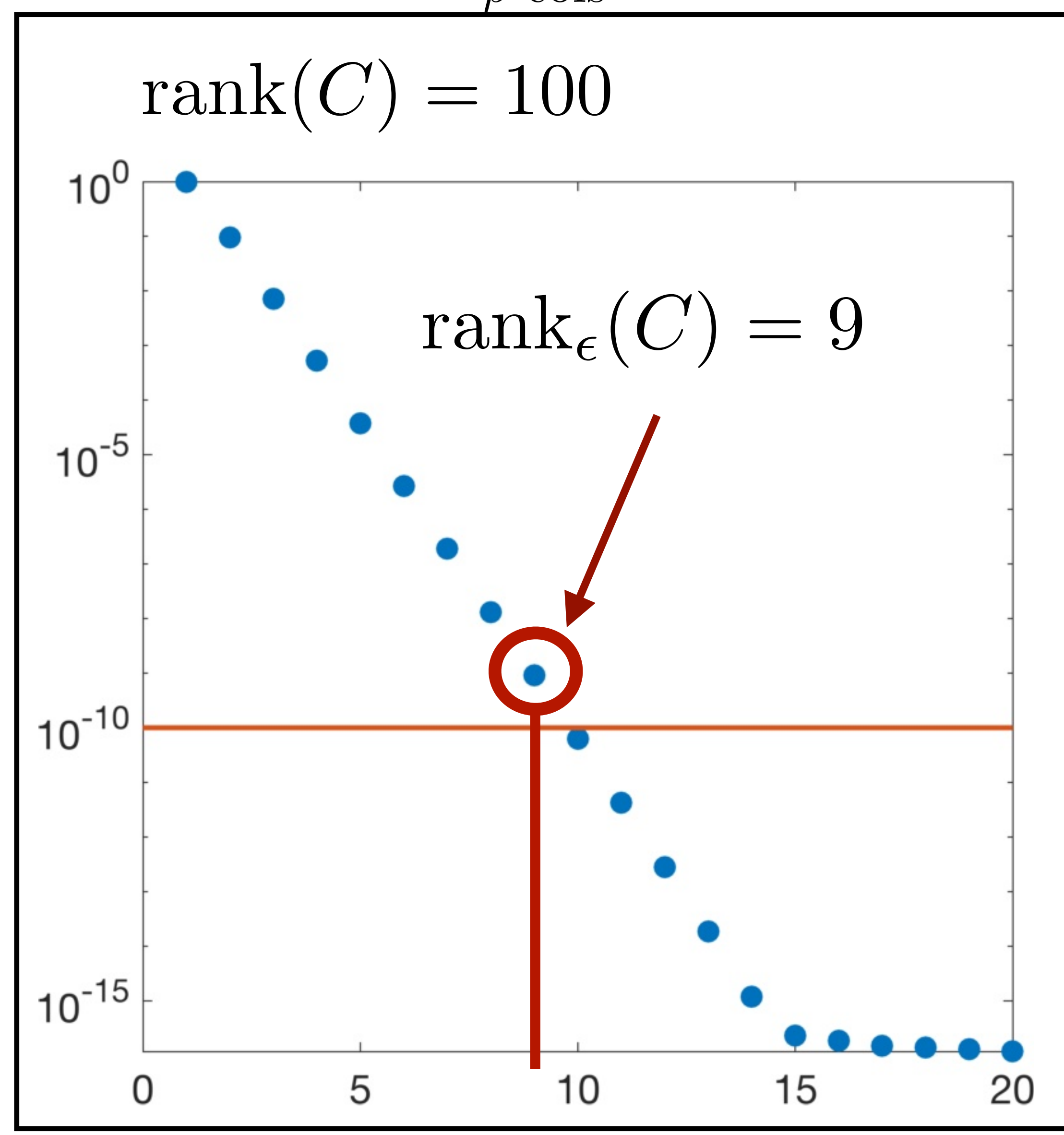
$$\operatorname{rank}_\epsilon(X) = \text{smallest } \rho \text{ where } \sigma_{\rho+1}(X) \le \varepsilon\|X\|_2$$



[Image from Ida, Nakashima, Kawai]



[Image from Ballani, Kressner]

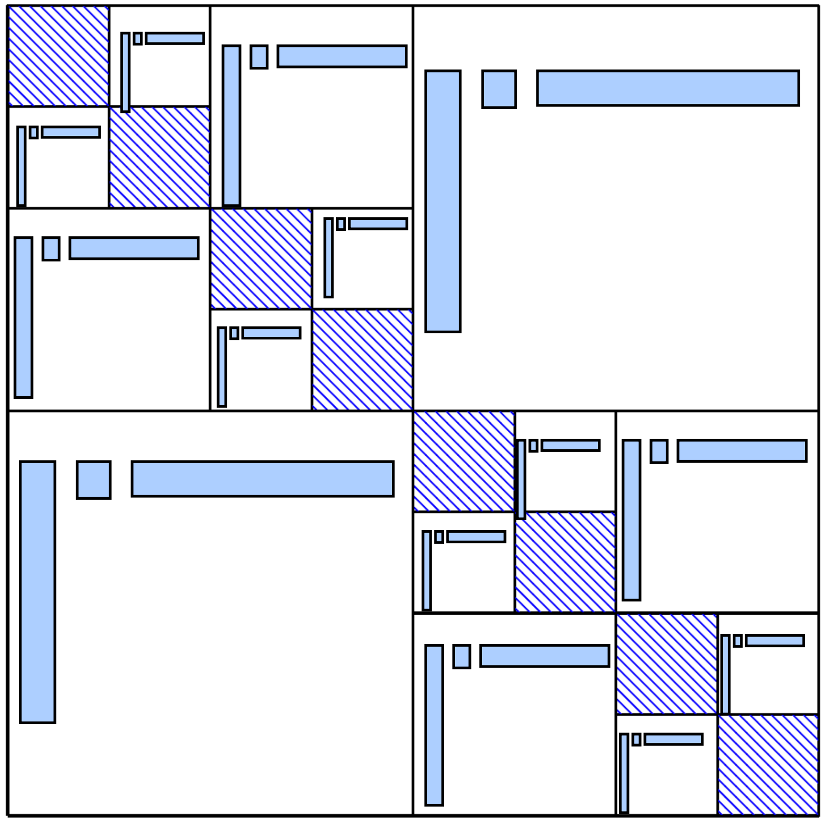



[Bebendorf, Chandrasekaran, Greengard, Hackbusch, Kressner, Martinsson, Rokhlin, Ying, Xia + many more]

# Rank-structured Toeplitz solvers

$$T = \begin{pmatrix} t_0 & t_{-1} & \cdots & \cdots & t_{-(n-1)} \\ t_1 & t_0 & \ddots & \ddots & t_{-(n-2)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & t_{-1} \\ t_{n-1} & \cdots & \cdots & t_1 & t_0 \end{pmatrix} \qquad \tilde{T} = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(n-1)} \\ t_{-1} & t_0 & \ddots & \ddots & t_{-(n-2)} \\ t_{-2} & t_{-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & t_{-1} \\ t_{-(n-1)} & \cdots & \cdots & t_1 & t_0 \end{pmatrix}$$

When $\tilde{T}$ is circulant, then $F\tilde{T}F^* = D$, with $D$ diagonal.
For general $T$, $FTF^* = C$.

$C \approx \tilde{C} =$ 

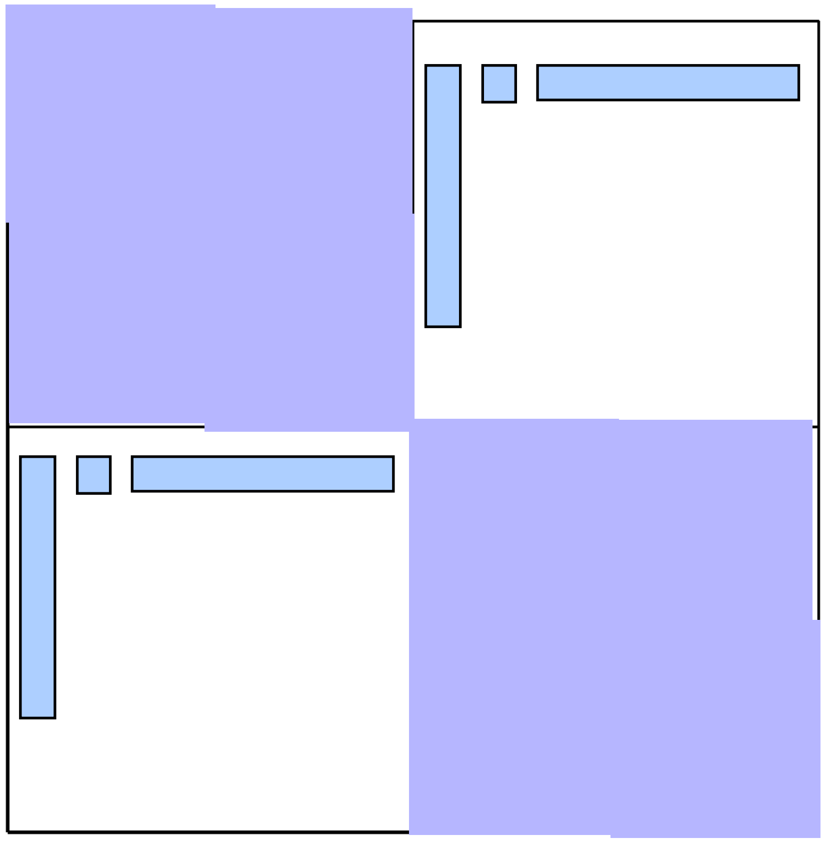
*A superfast solver for $Tx = b$*

1. Find $C = FTF^*$, $\tilde{b} = Fb$.
2. Find $\tilde{C}$, an approximate hierarchical factorization of $C$.
3. Solve $\tilde{C}\tilde{x} = \tilde{b}$.
4. Find $x = F^*\tilde{x}$.

[ Martinsson, Rokhlin, and Tygert (2005), Chandrasekaran, Gu, Xia, and Zhu (2007), Xia, Xi, and Gu

# Hierarchical representations: HODLR and HSS matrices

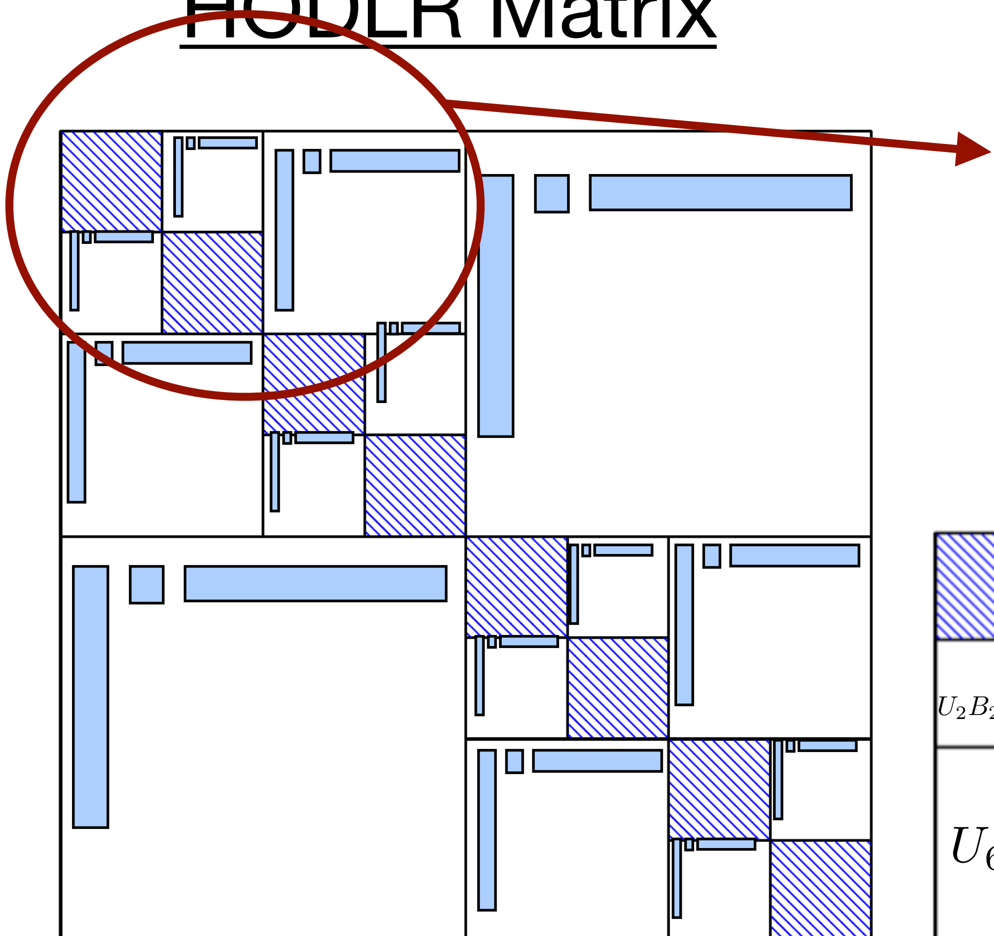
## HODLR Matrix



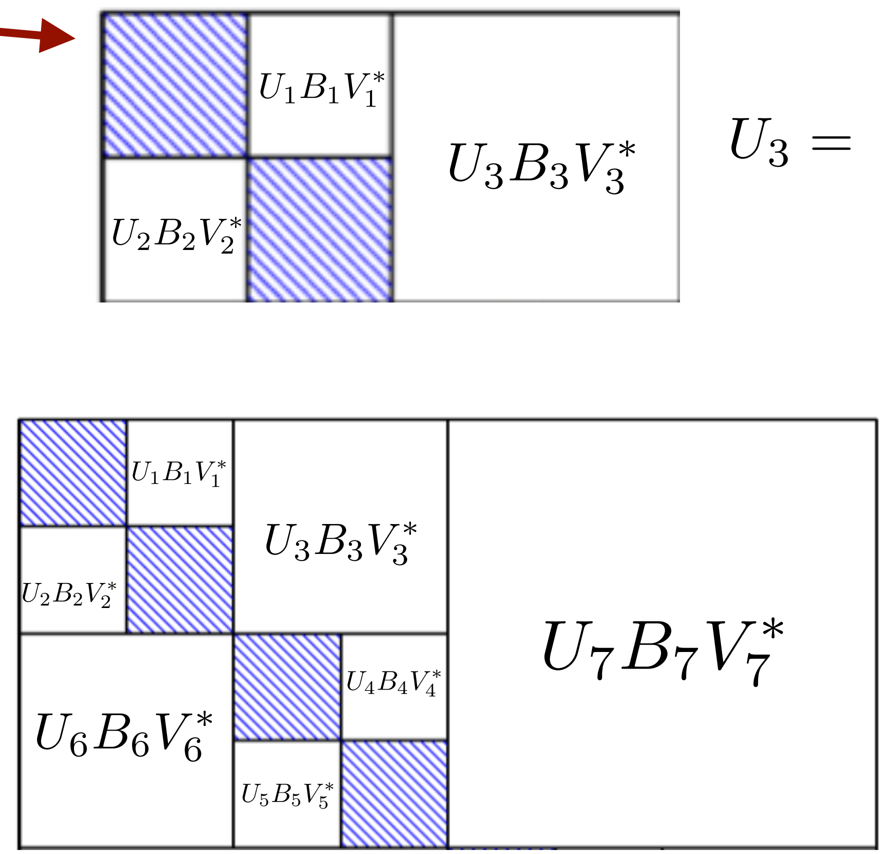
[Bebendorf, Chandrasekaran, Hackbusch, Kressner, Martinsson, Ying, Xia ]

# Hierarchical representations: HODLR and HSS matrices

## HODLR Matrix



## HSS Matrix




$$U_3 = \begin{bmatrix} U_1 & \\ & U_2 \end{bmatrix} R_3, \quad V_3 = \underbrace{\begin{bmatrix} V_1 & \\ & V_2 \end{bmatrix}}_{2m\times 2p} \underbrace{W_3}_{2p\times p}$$

$$U_7 = \begin{bmatrix} U_1 & & & \\ & U_2 & & \\ & & U_4 & \\ & & & U_5 \end{bmatrix} \begin{bmatrix} R_3 & \\ & R_6 \end{bmatrix} R_7$$

**HODLR**

$\mathcal{O}(np\log n)$ storage cost.

Solve $\tilde{C}\tilde{x} = \tilde{b}$ in $\mathcal{O}(p^3 n\log n + p^2 n\log^2 n)$ flops.

**HSS**

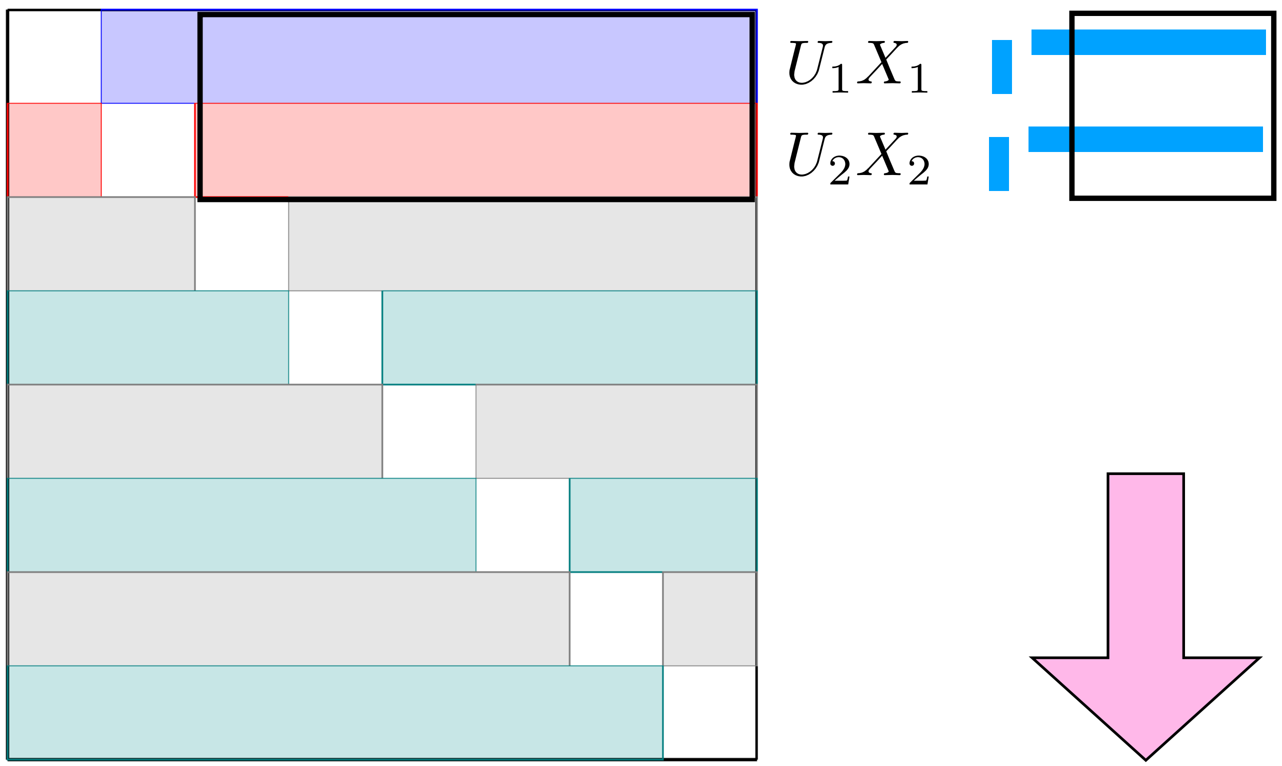
$\mathcal{O}(np)$ storage cost.

Solve $\tilde{C}\tilde{x} = \tilde{b}$ in $\mathcal{O}(np^2)$ flops.

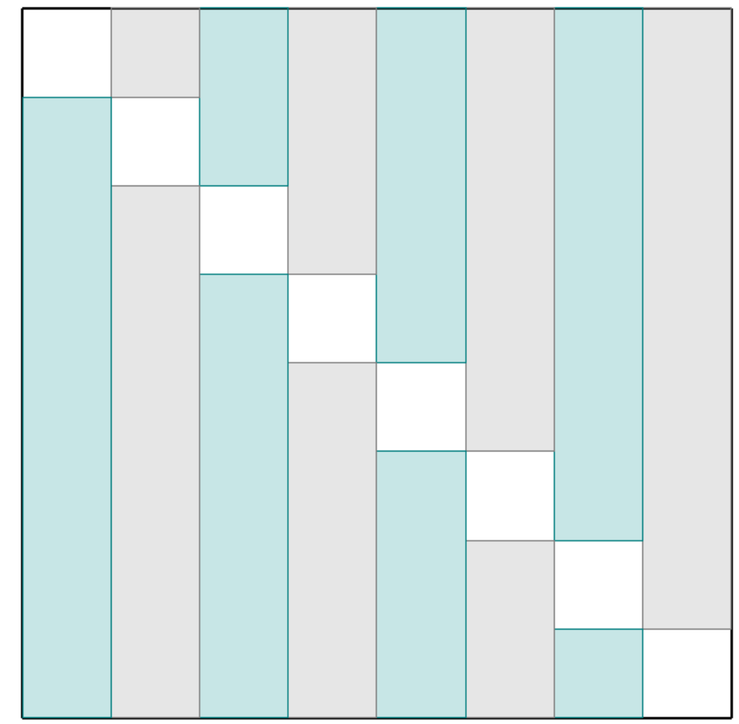
[Bebendorf, Chandrasekaran, Hackbusch, Kressner, Martinsson, Ying, Xia ]

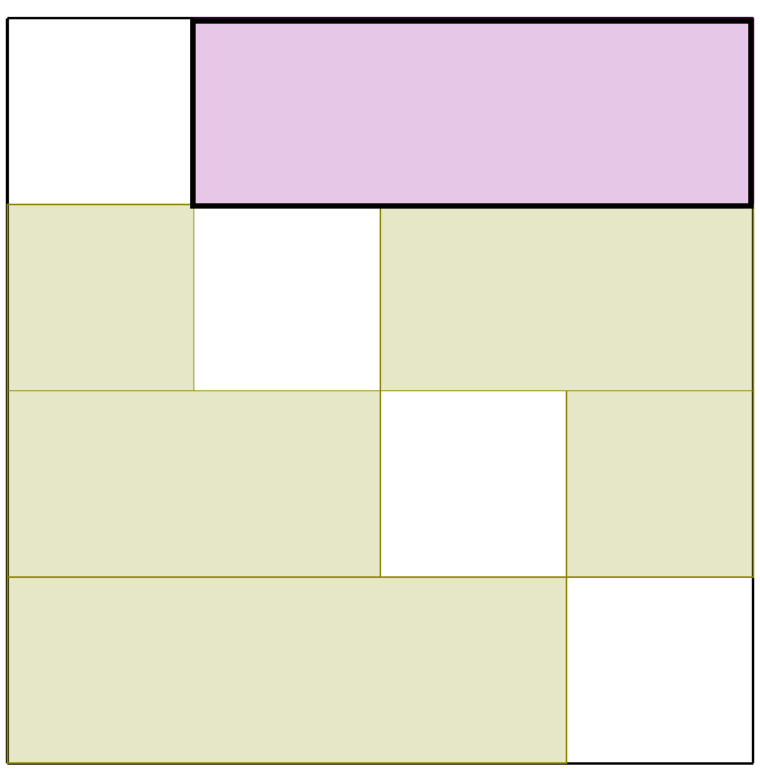
# Hierarchical representations: HSS rows and columns

HSS rows




HSS columns

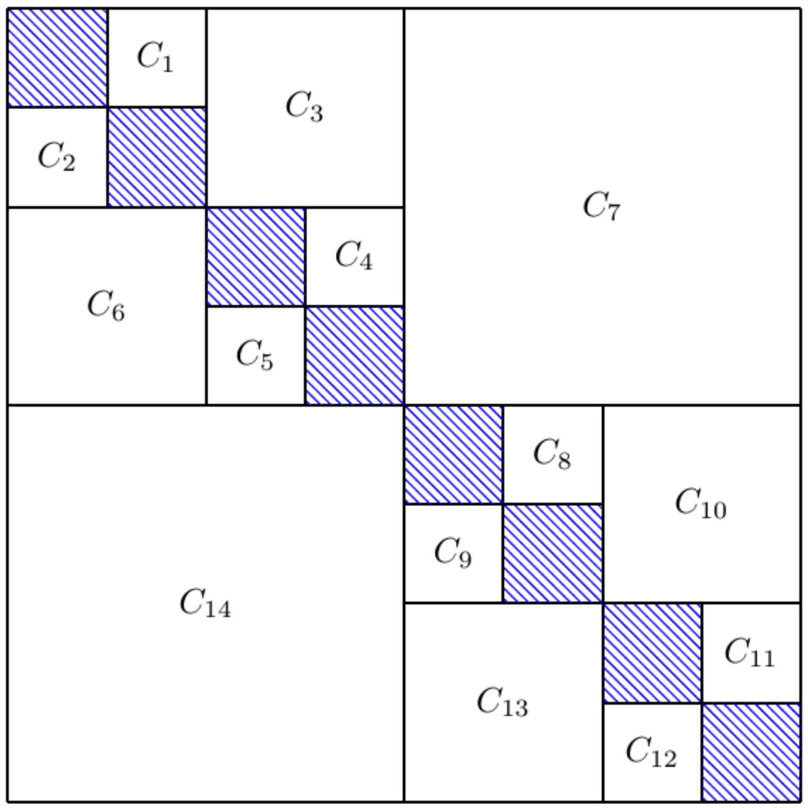


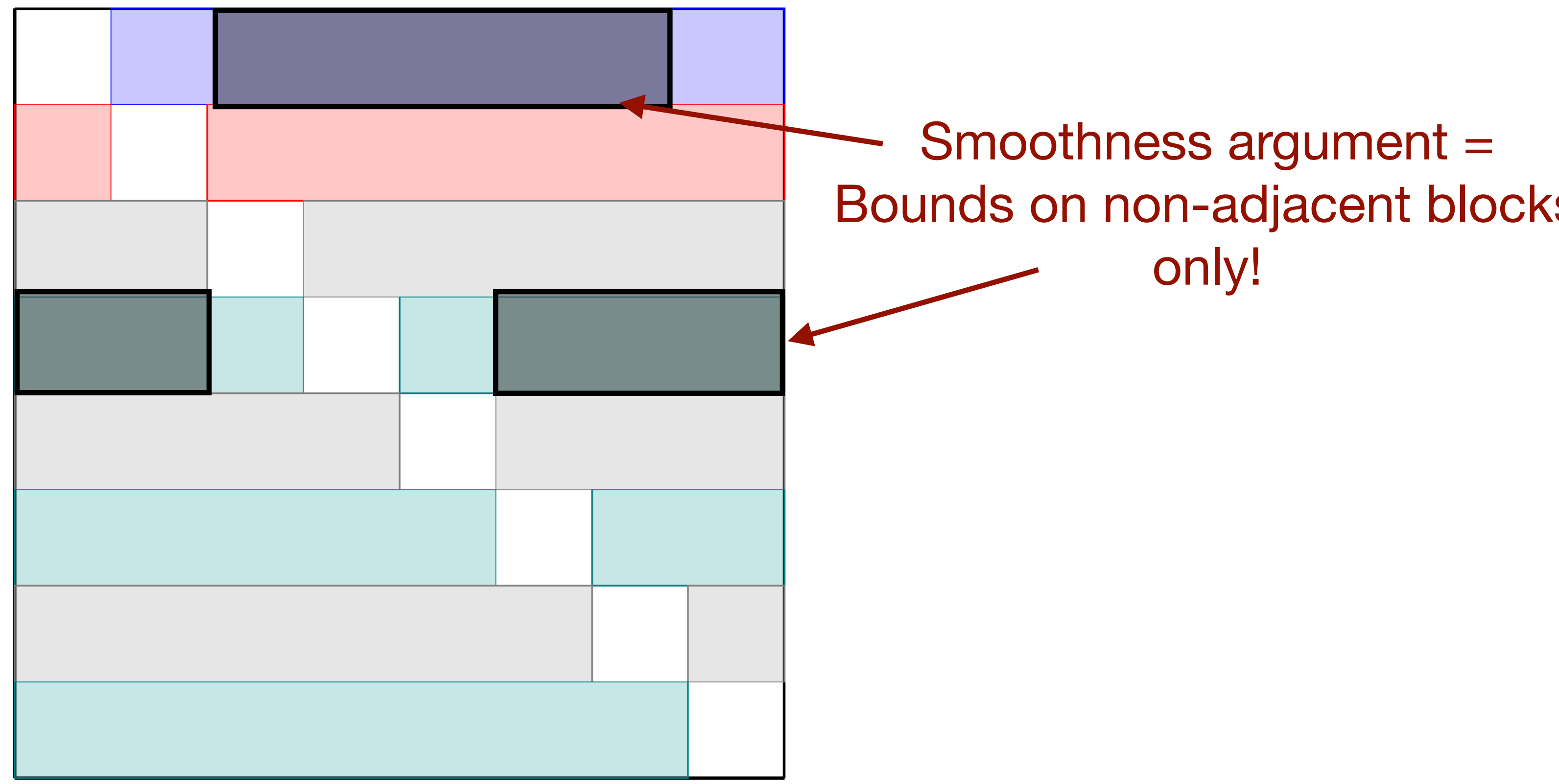


$$\begin{bmatrix} U_1 & \\ & U_2 \end{bmatrix} \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} = \begin{bmatrix} U_1 & \\ & U_2 \end{bmatrix} R_3 X_3 = U_3 X_3$$

# The rank structure of C

Goal: Bound the $\epsilon$-ranks of the submatrices of $C$.
Provide a recipe for finding low rank factors.




$$C_{jk} = \frac{a_j b_k + c_j d_k}{\omega^{2j} - \omega^{2k}}, \quad j \neq k$$

$$\omega = e^{\pi i / n}$$

Key Observation: We can use the displacement structure of C to build low rank approximations.

# Displacement structure and low rank approximation

$$AX - XB = F \qquad X \text{ has } (A, B) \text{ displacement structure}$$

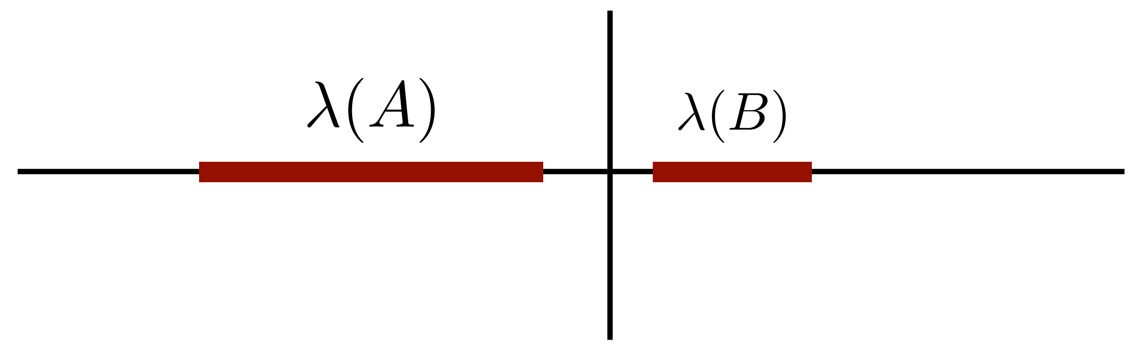
<u>Theorem</u> (Beckermann, Townsend, 2019):
If

1) $A$ and $B$ are normal matrices,
2) $A$ and $B$ have disjoint spectra,
3) $\text{rank}(F) \leq \rho$,

then

$$\sigma_{k\rho+1}(X) \leq Z_k(A,B)\|X\|_2.$$

When $\lambda(A), \lambda(B)$ are well-separated, $Z_k(A,B)$ decays rapidly.




$$\Longrightarrow \quad Z_k(A,B) \leq 4\eta_{cr(A,B)}^{-k}$$

$$C_{jk} = \frac{a_j b_k + c_j d_k}{\omega^{2j} - \omega^{2k}}, \quad j \neq k$$

$$\omega = e^{\pi i/n}$$

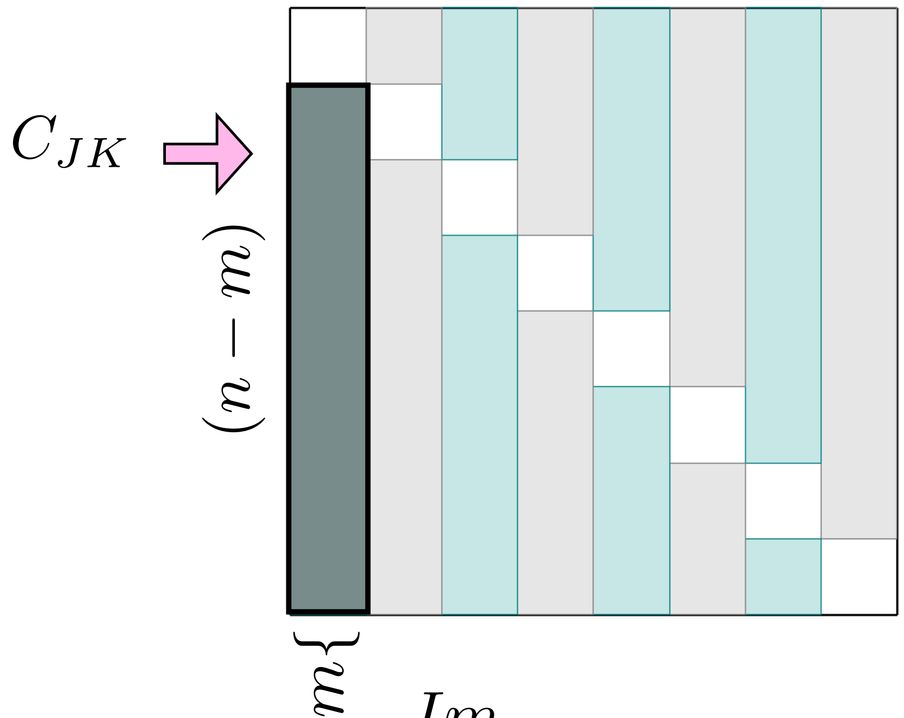
$$\Lambda C - C\Lambda = GH^*, \ \text{rank}(GH^*) = 2,$$

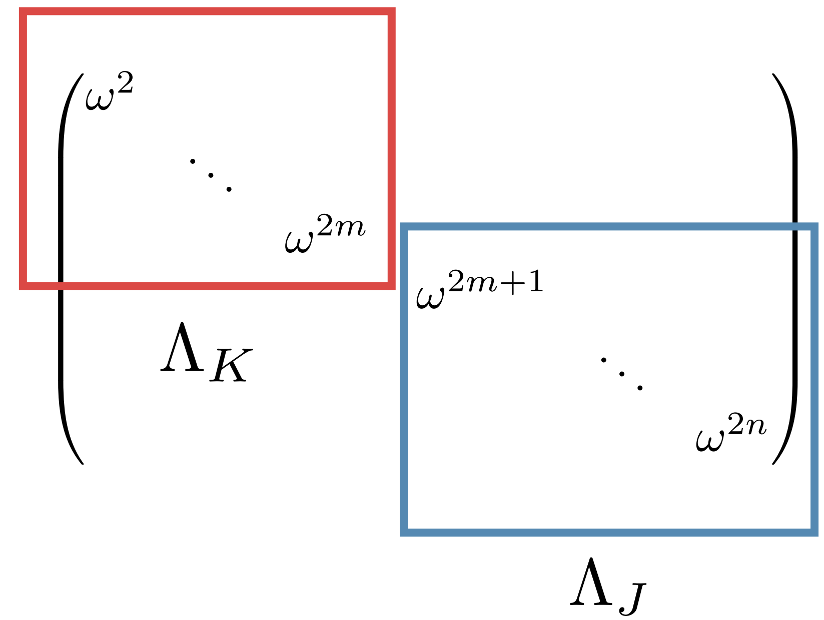
$$\Lambda_{jj} = \omega^{2j}$$

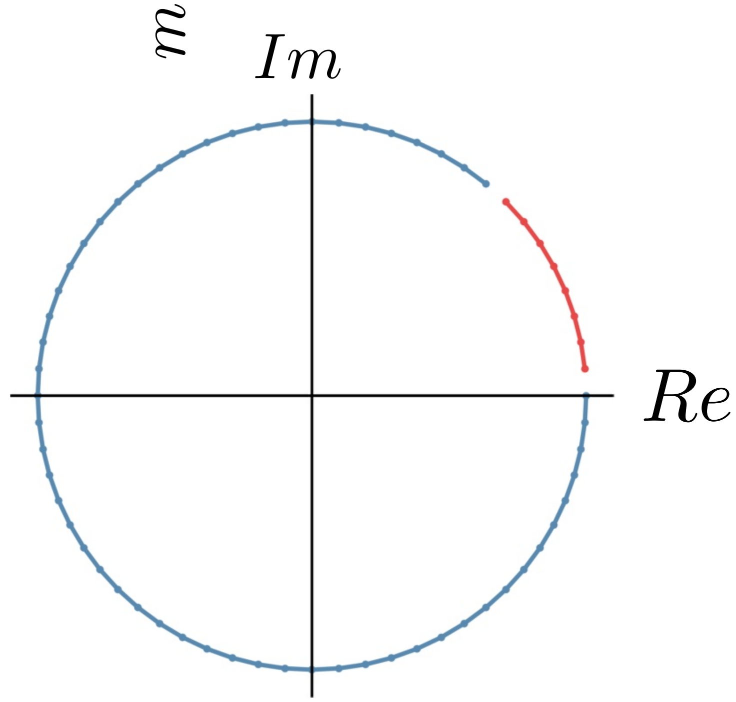
[ (Beckermann & Townsend, 2017), (Antoulas, Sorenson & Zhou, 2002), (Sabino, 2008 ), (Penzl, 1999), (Benner, Truhar & Li, 2009), (Li & White, 2002), (Druskin, Knizhnerman & Simoninci, 2011), (Peaceman & Rachford, 1955), (Lu & Wachspress, 1991), (Kressner & Sirkovic, 2015), (W., Townsend, & Wright, 2017), (Townsend & Fortunato, 2017) ]

# Displacement structure and rank structure in C

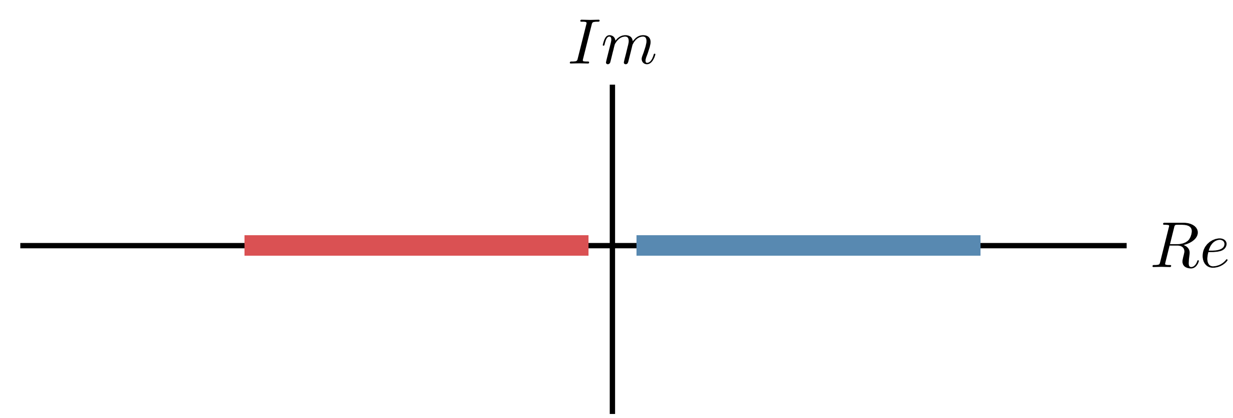
$$\Lambda C - C\Lambda = GH^*, \ \mathrm{rank}(GH^*) = 2, \qquad \Lambda_J C_{JK} - C_{JK}\Lambda_K = G_J H_k^*$$




$\Longrightarrow$




$$\sigma_{2\ell+1}(C_{JK}) \leq Z_\ell(\Lambda_J, \Lambda_K)\|C_{JK}\|_2 \implies \sigma_{2\ell+1}(C_{JK}) \leq 4\eta_{cr(\Sigma_J,\Sigma_k)}^{-\ell}\|C_{JK}\|_2$$

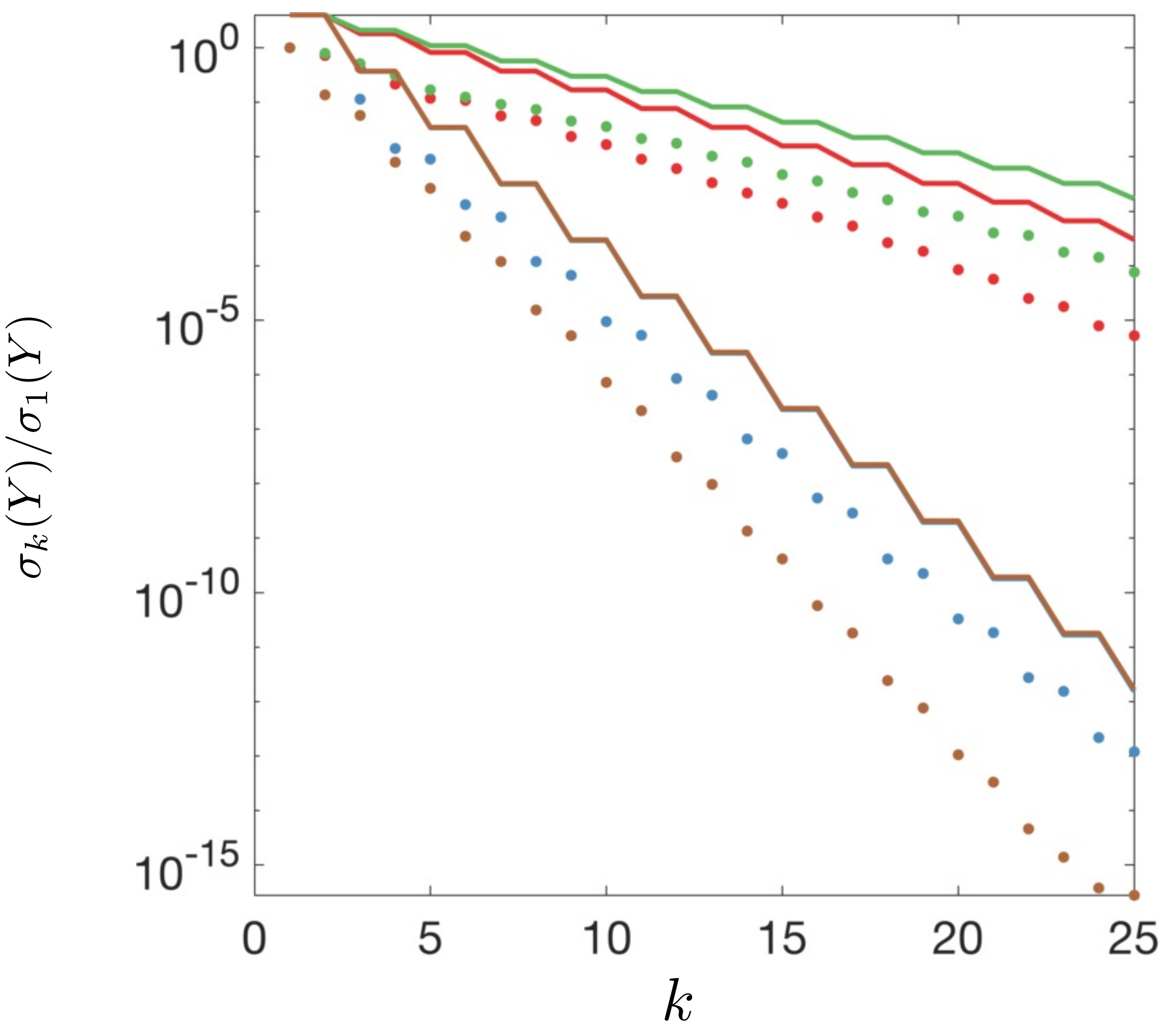
# The rank structure of C explained

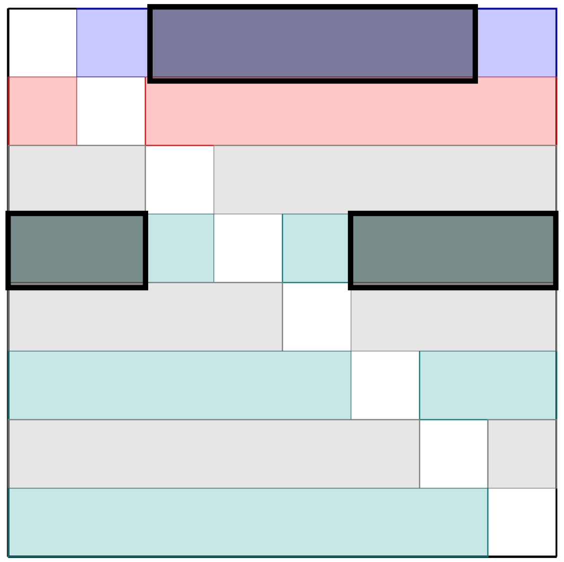
<u>Theorem</u> (Beckermann, Kressner, W.): If $Y$ is an 'adjacent' submatrix of $C$, then

$$\text{rank}_\epsilon(Y) \leq 2\left\lceil \frac{2}{\pi^2}\log(2n)\log\left(\frac{4}{\epsilon}\right)\right\rceil.$$

If $Y$ is a 'non-adjacent' submatrix of $C$, then

$$\text{rank}_\epsilon(Y) \leq \rho\left\lceil \frac{2}{\pi^2}\log(8)\log\left(\frac{4}{\epsilon}\right)\right\rceil.$$



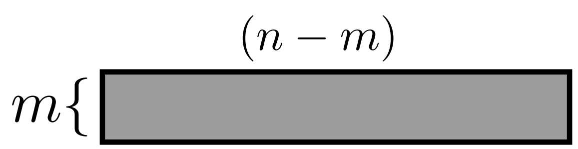



# A superfast, adaptive HSS-based Toeplitz solver

## factored ADI: A recipe for low rank approximations

[Benner, Truhar & Li (2009), Li and White (2002), Townsend, W., (2018)]

$$\Lambda_J Y - Y \Lambda_K = G_J H_K^*$$




$Y \approx Y^{(k)} = ZW^*$, $\text{rank}(Y^{(k)}) \leq 2k$
Finds $Z$ in only $\mathcal{O}(mk)$ operations.

$$\|Y - Y^{(k)}\|_2 \leq 4\eta_{cr(A,B)}^{-k} \|Y\|_2$$

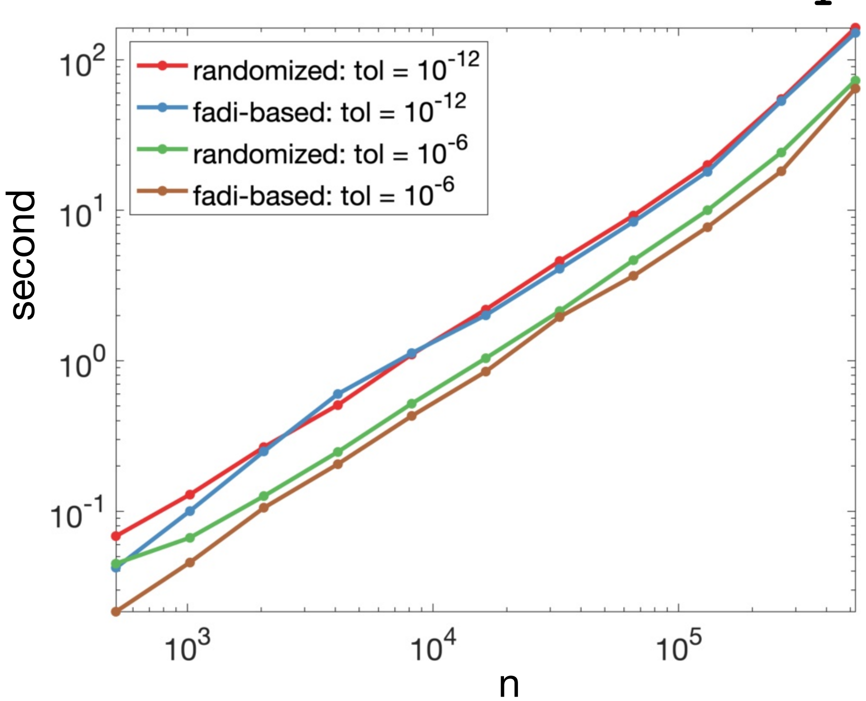
fADI-based interpolative decomposition of $Y$: $\mathcal{O}(p^3)$

Construction of HSS approximant $\tilde{C} \approx C$: $\mathcal{O}(p^2 n)$

Solving $\tilde{C}\tilde{x} = \tilde{b}$: $\mathcal{O}(p^2 n)$

$$p = \mathcal{O}(\log n \log \tfrac{1}{\epsilon})$$

```
x = Toeplitz_solve(trow, tcol, b, tol)
```




| tolerance setting | HSS construction error | Solution error |
|---|---|---|
| | $\|C - \tilde{C}\|_2 / \|C\|_2$ | $\|x - \hat{x}\|_2 / \|x\|_2$ |
| $10^{-3}$ | $1.887 \times 10^{-3}$ | $5.648 \times 10^{-3}$ |
| $10^{-6}$ | $4.567 \times 10^{-7}$ | $9.110 \times 10^{-7}$ |
| $10^{-9}$ | $3.623 \times 10^{-12}$ | $4.611 \times 10^{-11}$ |
| $10^{-12}$ | $6.445 \times 10^{-14}$ | $3.431 \times 10^{-13}$ |

# Summary

A Toeplitz matrix is only a fast transform away from a rank structured matrix…

Using displacement structure, we get

- thorough understanding of rank structure
- explicit approximation and error bounds
- cheap and easy compression/factorization strategy

| Superfast, adaptive solvers (code coming soon!) |
|---|

Other matrices that are just one fast transform away from a rank structured matrix:

Toeplitz-like
Toeplitz + Hankel
NUDFT
…

# Displacement structure and low rank approximation

$$AX - XB = F \qquad X \text{ has } (A,B) \text{ displacement structure}$$

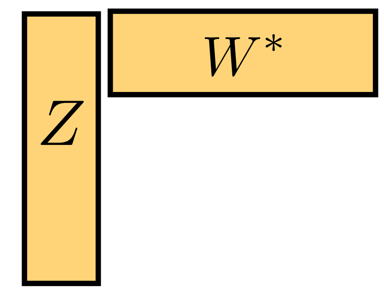
<u>(factored) ADI: A recipe for low rank approximations</u>

1. Solve $(A - \tau_{j+1}I)X^{(j+1/2)} = X^{(j)}(B - \tau_{j+1}I) + F$ for $X^{(j+1/2)}$.
2. Solve $X^{(j+1)}(B - \kappa_{j+1}I) = (A - \kappa_{j+1}I)X^{(j+1/2)} - F$ for $X^{(j+1)}$.

<u>After k iterations:</u>

- $X^{(k)} = ZW^*, \quad \mathrm{rank}(X^{(k)}) \leq k\rho, \quad \rho = \mathrm{rank}(F)$
- $X - X^{(k)} = r_k(A)Xr_k(B)^{-1}, \;\; r(z) = \prod_{j=1}^{k} \dfrac{z - \kappa_j}{z - \tau_j}$

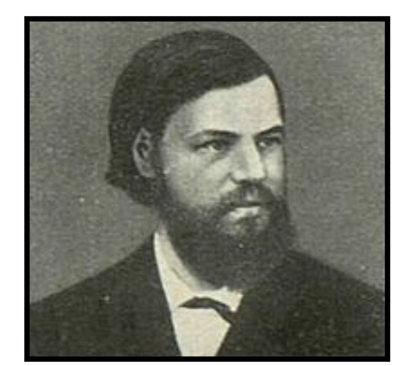
$X^{(k)} =$




$$\sigma_{k\rho+1}(X) \leq \|X - X^{(k)}\|_2 \leq \|r_k(A)r_k(B)^{-1}\|_2\|X\|_2 \leq Z_k(\lambda(A),\lambda(B))\|X\|_2$$

<u>Zolotarev's third problem:</u>

$$Z_k(\lambda(A),\lambda(B)) := \inf_{r\in\mathcal{R}^k} \frac{\sup_{z\in\lambda(A)}|r(z)|}{\inf_{z\in\lambda(B)}|r(z)|}$$



(Y. I. Zolotarev)

[ (Beckermann & Townsend, 2017), (Antoulas, Sorenson & Zhou, 2002), (Sabino, 2008 ), (Penzl, 1999), (Benner, Truhar & Li, 2009), (Li & White, 2002), (Druskin, Knizhnerman & Simoninci, 2011), (Peaceman & Rachford, 1955), (Lu & Wachspress, 1991), (Kressner & Sirkovic, 2015), (W., Townsend, & Wright, 2017), (Townsend & Fortunato, 2017) (Townsend & W., 2018) ]